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(Elementy Tochnykh Priborov)

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S. S. Tikhmenev

ELEMENTS OF PRECISION INSTRUMENTS

Handbook
for Calculation and Design

Approved by the Chief Administration
of Polytechnical and Machine-Building Higher
Educational Institutions of the USSR Ministry
of Higher Education, as a Textbook for Higher
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Government
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Moscow 1956

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S. S. Tikhmenev

ELEMENTS OF PRECISION INSTRUMENTS

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This book presents the fundamentals of precision instrument calculation and design, as required by students for design projects during courses and toward the diploma.

The first portion deals with various sources of error in instruments and certain methods of compensating and allowing for error. Much attention is given to the question of dynamic errors in instruments and particularly to the question of the effect of vibration on the functioning of instruments.

In the second part, methods of analysis of the individual mechanical parts of instruments are dealt with: elastic components, transmitting mechanisms, temperature compensation and the like. As far as possible, all such calculations are given in full.

Calculations for the electrical components of instruments are not presented in this book.

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PREFACE

This book treats certain problems arising in the calculation and development of instrument designs in industry and in course and diploma projects. These problems relate to electrical instruments chiefly in the sections in which examination is made of the total instrument error and the dynamics of the instrument mechanisms as a whole. Those portions of the book in which the individual mechanical elements of instruments are examined touch on electrical instruments only to the degree that particular mechanical elements are involved in their design.

Calculations for the electrical components of instruments are not derived, since they have been adequately covered in the literature (Bibl.1).

It should also be noted that the present work deals only with calculation problems common to various instruments or analogous mechanisms. Methods of choosing the principal elements of mechanisms and of analyzing assemblies and parts in a variety of instruments are offered. The influence of various factors on the functioning of a series of instruments is examined. Errors common to many different instruments are described, as are methods of adjusting instruments. However, the elements of no single instrument or mechanism are examined here unless there is reason to assume that similar components may find future application in other mechanisms. Thus, in treating the sensitive elements of instruments, we will pay attention primarily to elastic elements because, in view of the properties and function of sensitive elements, they usually differ completely from instrument to instrument, so that it is only with respect to elastic components that general

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observations may be made at all. In this connection, we do not offer a complete presentation of the methods of calculating the usual types of springs with linear characteristics (ratio of stress to deflection), since these questions are dealt with adequately in the appropriate sources.

Nor will we discuss the derivation of formulas for calculating flat and corrugated diaphragms and diaphragm chambers, in view of the fact that this is a highly complicated special question, a presentation of which would require a great deal of space. Furthermore, a special literature on this subject exists (works by V.I. Feodos'yev, D.Yu.Panov, L.Ye.Andreyeva and others, Bibl.2).

In related fields, we also do not derive here the formulas for calculating various types of sylphons and Bourdon tubes, since this has been adequately done in the works of V.I.Feodos'yev (Bibl.3).

INTRODUCTION

Section 1. General Data on the Classification of
Instrumentation Equipment

The devices termed instruments can be categorized as follows:

1. Measuring instruments;
2. Statoscopes (null instruments);
3. Integrating instruments;
4. Automatic controls.

In metrology, the term "measuring instrument" designates devices used for making comparisons between the magnitude being measured and the unit of measurement.

Statoscopes are instruments that do not offer quantitative data, but merely yield evidence as to the presence or absence of a given phenomenon. In an aircraft, for example, such instruments include the gyroturm indicator, the turn-and-bank indicator, statoscope altimeter formerly in use, etc.

Integrating instruments include those used for rapid solution of problems under certain special conditions, as for instance in flight. This may be required either to correct instrument readings or for complex measurements, i.e., when the value sought is obtained as the result of determining a number of initial magnitudes and subsequent calculations. Instruments of this type include slide rules, aircraft course correctors, etc.

Automatic controls include devices (often composed of several instruments) which exercise specific effects on other mechanisms without human assistance as a

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result of instructions transmitted by measuring instruments constituting a portion of the assembly. On an aircraft, for example, we find automatic pilots, boost controls, rpm controls, etc.

Measuring instruments can be subdivided into the following subcategories, in accordance with their method of use:

1. Direct-reading instruments;
2. Instruments requiring setting.

A direct-reading instrument is one that indicates data without requiring any intervention by the observer. Direct-reading instruments on aircraft include, for example, the compass, airspeed indicator, altimeter, fuel gage, etc.

Instruments requiring resetting include those which give readings only on supplementary displacement of the instrument itself or parts thereof. Included in this category, for example, are course-setting sights and sextants.

In addition, instruments can be classified by purpose. Thus, aircraft instruments can be subdivided into the following groups by nature of use:

1. Power-plant and propeller instruments;
2. Flight and navigation instruments;
3. Accessory instruments.

The power-plant and propeller instruments include those used for engine control: tachometers, pressure gages, vacuum gages (boost controls), air thermometers, fuel gages, propeller pitch governors, etc.

The flight and navigation instruments embrace those which assist the pilot in flying the aircraft: airspeed indicators, turn-and-bank indicators, chronometers, sextants, integrating instruments, etc.

Flight and navigation instruments, in turn, are subclassified into instruments installed only in the pilot's compartment, in the navigator's compartment, and in both.

Accessory instruments include instruments and automatic controls which are to

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create the required working conditions for the pilot and crew (oxygen and other instruments indicating the degree of flight safety).

Instruments for any other purpose may be classified in similar fashion.

Section 2. Major Components of Measuring Instruments

The theory of measurement states that measurements may be direct, indirect, or combined.

Direct measurements can be made only when a direct comparison of the magnitude to be measured and the unit of measurement can be made, the unit being rendered as a simple object constituting a material reproduction of the reference standard.

In all other cases, indirect or combined methods of measurement have to be used.

In indirect measurement, it is impossible to measure the magnitude of actual interest; instead another value, related to the first by a given regular relationship, must be measured. In the majority of cases, when this is the situation, the magnitude actually measured is a force or moment which ultimately deflects a pointer along the scale of an instrument.

Many direct-reading measuring instruments consist of the following three main parts:

1. The sensing element or pickup receptive to the physical phenomenon that the instrument is to measure.

When measurement is performed, some particular point on the pickup is displaced, the displacement being a specific function of the magnitude to be measured. This conversion of the variation in the magnitude being measured into a corresponding displacement of a given point or into rotation of a particular axis of a specific device may, in many cases, proceed through a number of stages. That is, it may consist of several successive conversions of one physical magnitude into another (i.e., electrical resistance, electromotive force, etc.).

When measurement is indirect, the parameter sensed by the pickup is in many

cases converted into a moment or force constituting a function of the magnitude to be measured, and this moment or force is equalized by some elastic force (usually the force of the measuring spring). This produces corresponding displacements of the points of action of the force.

In some instruments, both these functions are performed by a single pickup of unit design. In others, the conversion of the magnitude to be measured into a force, followed by balancing (measurement) through an elastic force, is done by various components of the instrument. In such cases, two or even more individual components are provided to carry out these functions of the pickup:

- a) a converter changing the magnitude to be measured into a force; if this is accomplished by several consecutive conversions of one physical magnitude into another, the converter may, in particular instances, consist of a number of individual components;
- b) an elastic element, balancing (measuring) this force.

2. The transmitting mechanism serves to transfer the displacement from the pickup to the pointer.

3. Scale and pointer (hand).

In many instruments, one of the components of the transmission mechanism is a so-called hairspring or catwhisker whose purpose is the following:

The transmitting mechanism has to effect a specific functional connection between the displacement of the pickup and that of the instrument pointer. For this, corresponding parts in sliding connections must be kept in constant contact. Where a chain drive is used, tension must be maintained on the chain. In addition, ball-and-socket joints always have a clearance between journal and bearing, and there is another gap between the teeth of a toothed-gear connection.

These factors may cause an impairment in the functional connection between the displacements of the two ends of the transmitting mechanism. To keep the STAT of the mechanism always in proper contact, a certain degree of unidirectional

tension must be imparted to the entire mechanism. This is done simplest by means of springs acting on the final unit in the mechanism (usually the axis of the pointer). Therefore, a hairspring ("catwhisker") is placed against the axis of instrument pointers.

As in most instruments, the full-scale deflection of the pointer is almost 360° (sometimes more); this purpose is usually served by a small spiral spring, one end of which is fastened to the pointer axis and the other to some part that is fixed relative to the body of the instrument.

Thus, the general pattern of operation of many direct-reading measuring instruments (and of the majority of mechanical instruments) is as follows:

The force T directly measured by the instrument, is generally some function $\Phi(N)$ of the magnitude N of the parameter to be measured. In a particular case, a linear relationship may exist between the quantities T and N , so that $T = CN$, where C is a constant.

The displacement of a point x of the pickup, to which the transmitting mechanism is connected (counting from the initial position of this point) is some function $f(T)$ of the magnitude of the force T and may be nonlinear in certain elastic components.

The displacement y (or the angle of rotation ϕ) of the pointer, counting from its initial position, is a function $\theta(x)$ of the displacement x , which is nonlinear in many transmitting mechanisms. Thus, in the general case, we have

$$\left. \begin{aligned} T &= \Phi(N); \\ x &= f(T); \\ y &= \theta(x). \end{aligned} \right\} \quad (A)$$

Therefore, the displacement of the pointer, in the final analysis, is some function $F(N)$ of the quantity being measured. It is often desirable that this function be of a given determinate character, and sometimes very specific in form

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(as, for instance, the equidistant spacing of an instrument scale). However, the form of this function depends on the forms of all three functions in eq.(A).

Under these circumstances, the form of the function $\phi(N)$ is usually fixed, since it is generally governed only by the operating principle of the instrument. Therefore, in order that the $F(N)$ function will be of the desired nature, the functions $f(T)$ and $\theta(x)$ must be selected accordingly.

Very often, proper selection of the $\theta(x)$ function of the transmitting mechanism may have a significant effect on the form of the $F(N)$ function. In the majority of cases, this selection of the $\theta(x)$ function is the major means of obtaining an $F(N)$ function of the desired nature. However, in some cases, selection of the $\theta(x)$ function in this manner may prove to be complicated and sometimes virtually impossible, if no attention is paid in the process to appropriate selection of the $f(T)$ function. Therefore, in designing instruments, an effort is usually made to select a pickup in which the $f(T)$ function is best suited to the case. Sometimes the pickup is modified for this purpose.

In most springs contained in the mechanisms of instruments for measuring concentrated forces, the deflection of the pointer, within the interval of permissible stresses, is almost directly proportional to the load applied to the spring. A similar situation exists in certain spring-loaded elements used for pressure measurement. Thus, the free end of a Bourdon tube moves in a manner approximately proportional to the pressure differential between the inside and outside of the tube. The same may be said of certain siphons working in a given limited range of pressures to be measured. The use of similar elastic elements as instrumental pickups only permits a variation in the total range of forces measurable by the pickup by providing the element with the necessary degree of rigidity. But in this case the nature of the $F(N)$ function cannot be changed in the required direction. Therefore, if the resultant traverse x of the pickup is of such a nature as to be

g a specific pattern at the instrument scale, all that need STAT

be done is to select the appropriate transmitting mechanism.

Pickups also exist in which the traverse is not proportional to the force being measured (as, for instance, in diaphragm chambers). In certain cases, this complicates the further selection of the transmitting mechanism, but sometimes, on the other hand, may even simplify it. This depends on the nature of the function $T = \phi(N)$. In some special cases, the functions $\phi(N)$ and $f(T)$ are individually nonlinear, which greatly complicates the problem of obtaining a convenient presentation of the instrument scale. However, in combination they yield an $F(N)$ function that is sufficiently practical. An example of this is the altimeter, whose pickup consists solely of aneroid chambers, with no added springs. In the general case, however, this successful combination of the functions $\phi(N)$ and $f(T)$ facilitates matters only to a limited degree. Therefore, in order to obtain an $F(N)$ function of the desired nature, it is often necessary to select and vary certain parameters of the pickup, affecting the nature of the $f(T)$ function. This would hold, for example, for the type of bellows or diaphragm corrugation or for certain relationships between the dimensions of particular parts in the mechanism of the instrument. Sometimes in cases such as this, a variation of the $f(T)$ function is obtained by using accessory devices, which will be discussed below.

In practice, various deviations from this principal layout are encountered. These are due to special conditions of measurement. For example, a large percentage of instruments requiring "setting" before use do not have a pickup that automatically converts the phenomenon to be measured into a force, a moment, or some other physical value. In a course-setting sight, for example, the crosshairs and bubble level, acting as the pickup, do not react automatically to changes in the quantity to be measured, as is the case for direct-reading instruments. In order to obtain a reading with the navigation pendant the crosshairs must be adjusted relative to the other components of the instrument.

Certain direct-reading instruments have no transmitting mechanism. This is

true for the electric measuring instruments used in electric tachometers, thermometers, etc. This is explained by the fact that the moments developed by the pickups are very small (of the order of 0.2 to 0.1 gm-cm).

Let us note one more type of instrument, the so-called remote-indicating instrument, quite frequently encountered in aviation.

The use of instruments of this type is due to the necessity of measuring quantitative factors or various phenomena occurring at a distance from the observer. Remote-indicating instruments usually consist of three parts:

1. The responsive element, usually called the "sensing unit" or "pickup", which responds to the phenomenon to be measured or to a secondary phenomenon functionally correlated with that to be measured. The responsive element serves simultaneously as a converter for transforming the picked up phenomenon into some force or some other physical quantity (such as an electric resistance, etc.), whose magnitude depends on the value of the factor to be measured.

The aerodynamic airspeed indicator is an example of such an instrument. The pickup of this instrument converts the measurement of airspeed into the measurement of the pressure of the relative circulating air flow. Similarly, in certain electric tachometers, the pickup, consisting of a small generator, converts the measurement of the rate of rotation into measurements of the voltage, power, or frequency of the current generated as its armature revolves.

2. The transmitter, serving to transfer the phenomena developed in the pickup to the measuring portion of the instrument. This is what happens, for example, to the pressure developed in the pickup of an airspeed indicator, the voltage or frequency of the current developed by the generator of an electric tachometer, etc.

Frequently, the transmitter unit consists simply of a pipe for transferring pressure or of an electric wire. In some cases, however, a more complex transmitter is required.

3. The measuring element, determining the magnitude of the quantitative factor

of a phenomenon developed in the pickup. The scale is laid out in units of the quantity to be measured by means of the given remote-indicating instrument.

In certain individual cases, remote-indicating instruments require numerous components.

In the present work, the instruments chiefly investigated are those of the direct-reading type as well as statoscopes many of which hardly differ in principle from direct-reading measuring instruments.

Section 3. Factors Entering Into the Determination of the General Design Pattern of an Instrument

In any measurement, the numerical value of the quantity to be measured differs to a greater or lesser degree from the true quantity, which remains unknown in the long run. The lack of agreement between the numerical value obtained as a result of measurement and the true value is the error of measurement.

The major function of each observation consists in making certain that the measurement will be of the required accuracy, i.e., in reducing the error of measurement to permissible levels.

The imperfections of instrumentation and of the measuring methods themselves often greatly complicate the work of the observer and make it necessary for him to know all factors that may constitute sources of error. An even greater knowledge of all possible error sources in the instrument itself is required of the designer. Only when all possible errors have been given full consideration will the designer be in a position to select a rational design for the instrument and choose the most desirable design parameters.

It should also be noted that when instruments are made in series, individual parts often differ widely. This is true primarily of the elastic components and particularly of the pickups. Pronounced irregularities are observed in pressure chambers, both as to their general mean elasticity and as to their characteristics. Furthermore, even ordinary springs in the same series are far from identical in

mean elasticity and characteristics.*

From this it follows that, in the design of the majority of instruments, it is essential to provide fairly broad possibilities for the adjustment of their mechanisms, or in other words, for changing the values of the various parameters. Strict observance of this requirement is particularly important since some parameters of the instrument mechanisms undergo a certain degree of change on their own, with the passage of time.

The conditions under which an instrument has to operate and the specifications as to weight and dimensions are important factors, affecting the design of an instrument and often even the choice of the physical principle on which it is based. Quite often these conditions require that the specifications for accuracy be somewhat reduced since, under the given conditions, they may be difficult or even impossible to fulfill.

The present book offers an analysis of various common error sources for which allowance must be made in instrument designing. In addition, examples of the conditions under which certain types of instruments have to operate and data as to the effect of these conditions on the choice of design are cited. The common methods of calibrating instruments will also be treated briefly.

One more highly significant factor that must be taken into consideration in choosing the general layout and developing the design of an instrument is the useful life of the design in terms of production requirements and in terms of economy. It is most important that the designed instruments consist of assemblies and parts that are easy to manufacture and assemble and subsequently to calibrate. In addi-

*By mean elasticity is meant the ratio of the total deflection of the elastic element in the range of exerted stresses (i.e., from minimum to maximum load) to the difference between these extremes. This characteristic is known as the functional ratio of the stress on an elastic element to its deflection.

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tion, instruments must, whenever possible, be inexpensive to make and to maintain. However, these questions are treated in greater detail in books on instrument construction engineering (Bibl.4 and 4a).

Let us note in passing that simplicity of design is a significant factor in more than the production sense and is important as a matter of principle. It is common knowledge that a simpler design operates more dependably than a complicated one and that it is difficult to adjust a complex instrument to operate properly within the assigned limits. This factor must never be disregarded in choosing the basic layout and in developing the design of an instrument.

Dependability in operation is one of the most important requirements that an instrument must meet. The instrument must function without failure and must not go out of adjustment for a prolonged period of time, under normal operating conditions. This requirement must be taken into consideration, both in selecting the basic principles and in developing the instrument, as well as in manufacturing them.

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PART ONE

INSTRUMENT ERROR, ITS DETERMINATION, AND SOME METHODS
OF COMPENSATION

CHAPTER I

ERRORS IN STATIC MEASUREMENTS

Section 1. Effects of the Operating Conditions of Instruments
on the Decision as to Design and on the Reading Accuracy

The operating conditions under which an instrument must function are often of decisive importance in selecting the design and often require a reduction in the specifications for accuracy.

As an example, let us cite the operating conditions of aircraft instruments under flight conditions and certain special specifications for these instruments, based on these conditions.

1. The range of temperature fluctuations in the environment of aircraft instruments is very wide. As a result, temperature errors arise which require a lowering of the accuracy specifications to be met by this category of instruments, as compared, for example, with laboratory instruments. In addition, it is often necessary that an instrument be provided with temperature compensation to reduce the magnitude of this error. In many cases, this condition affects the choice of design and even the physical principle employed in the instrument, since a certain principle (or design) may permit a smaller temperature error or an easier compensation than does another.

2. The range of fluctuations in external pressure experienced by an aircraft ^{STAT} is quite considerable. The functioning of many instruments depends little or not at all on this factor. However, for certain instruments, external pressure plays a

decisive role and affects the choice of design. Examples in this respect include airspeed indicators, air thermometers, various types of temperature compensation in compasses, etc.

3. Aircraft instruments have to meet higher specifications for strength, since they must remain properly adjusted in the face of possible shock and under the effects of vibration. In addition, it must be considered that under conditions of flight or of the mounting of instruments in aircraft, the same careful handling as in the laboratory is impossible. This may also affect the choice of instrument design and, as a result, the accuracy specifications.

4. Aircraft instruments must be as small and light in weight as possible, because of the fact that they are to be mounted on aircraft. This requirement considerably complicates the work of the designer and sometimes also affects the accuracy of the instrument.

In addition, a reduction in the sensitivity of the instrument (diminution of the scale) increases the reading error, while a smaller mechanism is more difficult to manufacture accurately. Theoretically, any reduction in the instrument dimensions leads to an increase in the error, due to after-effect and hysteresis. To prove this fact all that is needed is a comparison of two formulas for any type of spring: the elasticity of the spring and the maximum strain in the spring material, corresponding to maximum stress.*

The larger the dimensions of a spring, the easier will it be (without changing its elasticity) to select its individual dimensions in such a fashion as to reduce the strain in the spring material, at equal stress. From this it also follows that

*See any technical handbook, such as "Huetete", "Spravochnik metallista" (Metalworkers' Handbook), etc., for formulas for calculating cylindrical springs working under torsion, formulas for straight and cranked springs working under flexure, etc.

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a spring of larger dimensions is easier to build with greater flexibility per unit stress, without increased strains in its material. A reduction in the rigidity of a spring, however, or, in other words, an increase in flexibility, often has a significant effect in reducing the errors due to friction, in connection with a possible reduction in the transmission ratio of the mechanism.

5. Airborne instruments function under conditions of extensive variations in position and considerable accelerations. In order for these phenomena to have the least possible effect on the instrument readings, each individual part must be balanced as accurately as possible, since the axis of rotation of the various parts of a mechanism are usually not parallel, and the angles of rotation of the various parts are not identical during changes in instrument indications.

6. Instrument pointers should not have noticeable oscillations due to aircraft vibrations. Since the direction of perturbation forces changes rapidly under vibration, instrument pointers show noticeable oscillations only in the case of resonance, i.e., when the natural frequency of oscillation of the mechanism of the instrument is close to the frequency of vibration. Therefore, in designing aircraft instruments, it is essential that the natural frequency of vibration of their mechanisms differ sufficiently from the possible vibration frequencies of the aircraft (the latter usually being taken as equal to the rpm of the engine).

7. Aircraft instruments must be provided with scales that are precise and easy to read. This requirement is based on the conditions under which a pilot and navigator must work, particularly under military conditions, when their attention must be diverted as little as possible.

Section 2. Brief Classification of Instrument Errors.

The Main Types of Common Instrument Errors in Static Measurements

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The errors in measuring instruments can be subdivided into two groups:

- a) Partial errors, constituting peculiarities only of the given method or

the given instrument design;

- b) General errors, common as a rule to the majority of measuring instruments built to a standard design.

Partial errors are subject to investigation only on examination of the theory and specific design of a particular instrument. As far as common errors are concerned, their basic principle can be generalized and examined independently of a specific instrument design.

Furthermore, errors in instruments can be subdivided into three basic groups:

1. Methodical errors;
2. Instrument errors;
3. Personal errors (or reading errors).

Since personal errors are related to only a single design factor, namely the sensitivity of the instrument (scaling), we will limit the discussion to a brief comparison of these errors with instrument errors.

Methodical errors result from the defects in the given method of measuring a particular quantity. Most frequently these errors occur in indirect measurements, i.e., under conditions in which the factor for whose measurement the instrument is designed is not measured directly but via some other intermediate factor, having a specific functional relationship to the former.

Indirect methods of measurement such as this have to be used quite often. This being the case, there are some situations in which the intermediate factor, directly measured by the instrument, is a function not only of the first factor but also of other quantities. The instrument errors resulting from these causes are known as methodical errors. It is obvious that they differ from one type of instrument to the next. Therefore, we will limit the discussion to a general characterization.

Instrument errors are those resulting from defects or shortcomings of the design or manufacture of an instrument. These errors can be divided into errors of static and of dynamic measurements, known as errors of inertia or dynamic errors.

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The first category of errors includes those created regardless of the inertia of the moving parts of the instrument. The present Chapter is devoted to their investigation.

For a number of types of instruments, the following common causes of such errors exist:

1. Inaccurate Adjustment of the Instrument. This may be due to the following reasons:

- a) Residual error after regulation or calibration of the instrument;
- b) Changes in certain parameters of the instrument in time (for example, deformation of a spring due to fatigue).

This error belongs to the category of systematic errors and may, to some degree, be allowed for in the certificate accompanying the instrument.

2. Errors of Elastic Aging and Hysteresis. Elastic aging and hysteresis take place in instruments in which a force is measured by means of a spring or membrane. The effect of this factor on instrument readings is common knowledge.

Elastic aging and hysteresis belong to the category of accidental error, or in other words errors that can not be allowed for. In fact, the magnitude of aging depends on the total time during which the instrument has operated and on the type of readings it has recorded. Under actual service conditions, this is usually very difficult and often impossible to compute. Moreover, the degree of elastic aging and hysteresis is very inconstant, depending on temperature, vibration, and other causes which are often difficult to calculate.

To reduce the errors of elastic aging and hysteresis, the design of an instrument must not permit marked strains in its elastic components. However, as shown in Section 1 of the present Chapter, any reduction in the dimensions of the instrument greatly complicates this problem.

3. Errors of Friction. Friction in instrument mechanisms may vary in origin:

- 1) Occasionally, mechanical friction is so great that its results can hardly

be attributed to instrument errors and are more readily traceable to defects or trouble in the instrument. This type of friction effect is expressed as follows: The pointer either deflects in jumps or fails, by a considerable margin, to give a correct reading after changes in the magnitude of the force being measured, since this force had undergone a smooth change. In this kind of situation, a tap on the instrument will often result in the needle jumping to a more correct reading.

These phenomena may result from the following causes:

- Poor assembly of the instrument, such as skewing of lever-to-shaft connections, which sometimes occurs when a lever has been bent in adjusting the instrument;

- Jamming of parts due to expansion with temperature, if adequate clearance was not provided between meshing parts during assembly;

- Various defects in parts or their fouling;

- Inadequate or, at times, excessive strain on the hairspring, since extreme tension will cause the spring to kink. The hairspring must be so coiled that, under the maximum possible tension at various instrument readings, all turns in its spiral lie virtually in a single plane.

2) Stagnation, or common friction error. Stagnation in instruments, shows in the fact that an instrument gives different readings in measuring an identical quantity (in this respect, stagnation has the same manifestation as hysteresis). The difference in the readings depends on whether the given reading is approached from below, at a constant increase in the quantity being measured, or from above, at a constant decrease in the quantity.

The phenomenon of stagnation results from the fact that the forces activating the parts of the mechanism are inadequate to overcome the entire frictional drag in the instrument.

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If the functioning of the instrument were not subject to vibrations or shocks of any kind, all instruments containing rubbing parts would theoretically be subject to some degree of stagnation (although in some instruments this stagnation is so small that it remains virtually unnoticed). Actually, displacements in the mechanism of a working instrument result from the fact that the force being measured does not equal the counteracting force of the spring. However, a portion of the forces in action is expended on overcoming the friction in the mechanism. Therefore, when the difference between the force being measured and the force of the spring is equal or less than the combined force of resistance to friction in the mechanism of the instrument, displacements of the mechanism can no longer occur.

Stagnation is therefore qualitatively determined by the force required to overcome friction in the mechanism of an instrument.

It should be mentioned that, in a majority of aircraft instruments, stagnation is considerably reduced as a result of aircraft vibration.

Stagnation may usually be detected in the following manner while checking an instrument: Tapping the housing or dial will cause the pointer to move after the force to be measured has come to a standstill.

It is quite obvious that stagnation also applies to accidental instrument errors, i.e., those that cannot be foreseen or allowed for.

It is clear from the above statements that, in selecting the design pattern of an instrument, the layout with the lowest friction work in relative displacements of parts of the mechanism must be selected. A comparison of certain transmitter connections in this respect will be presented in the Section on transmitter connections.

In addition, a reduction in errors of friction, in designing an instrument, requires proper selection of the corresponding pickup. The designer must always consider the "efficiency" of the pickup he selects. In using this term, we refer to the product of the range of force developed by the pickup at the point of coupling

to the transmission, multiplied by the total displacement of this point from the minimum to the maximum reading of the instrument. This is quite obvious. The greater the forces being measured by the instrument, the smaller will be the ratio of the total friction forces applied to the pickup in the mechanism to the force developed by the pickup itself. In addition, the greater the traverse of the pickup, the less will be the total transmission ratio required and, consequently, the smaller will be the total force of friction in the transmitting mechanism applied to the pickup. We know that forces applied at any point in a given mechanism are applied to another point by multiplication by the ratio of the rate of displacement of the point at which the force is applied to the corresponding rate of displacement of the point to which this force is transferred.

In selecting the pickup of an instrument, two methods are available for reducing the effect of friction in the mechanism on the instrument readings:

Increasing the forces acting in the pickup (for diaphragm elements, this means increasing the effective area of the diaphragm, i.e., increasing its diameter);

Increasing the traverse of the pickup to reduce the transmission ratio of the mechanism (in diaphragm instruments, this can be done by increasing the number of consecutively connected chambers).

3) Errors due to clearances. What is known as backlash in an instrument shows in the fact that the instrument yields slightly different readings after being tapped repeatedly.

This phenomenon occurs under conditions in which a noticeable clearance appears in some coupling of the mechanism, due to a specific fault in manufacture. In a hinged joint, for example, a large clearance may have been left in the direction of the axis of rotation, with the result that the axis is not perpendicular to the attached link.

The hairspring of an instrument is of sufficient strength to take up all slack

in the direction in which the spring is applied to a given connection. It may, however, prove if inadequate strength to move a hinge on its axis, if the direction of the axis makes a large angle (close to 90°) with the direction of the link. Since the hinge is thus misaligned, various positions of the link will affect the positions of all subsequent parts of the mechanism, up to and including the pointer.

The same result may follow when there is a large clearance along the axis of a gear in a toothed connection, if the teeth of the gear are slightly beveled.

4. Errors due to Temperature. The term "temperature errors" applies to changes in the readings of an instrument on any fluctuation in the ambient temperature.

In instruments with an elastic pickup, the major cause of temperature errors is a change in the modulus of elasticity of the material from which the spring (or diaphragm) of the pickup is made, with any change in temperature.

In certain instruments, other specific reasons for temperature error may exist, depending on the operating principle and the design. However, this is not one of the subjects discussed in the general portion of this book.

One of the causes of temperature error may also be that parts were made of materials of different thermal expansion. Naturally, if all parts were made of a single material, temperature fluctuations would result in proportional changes in all their dimensions and would not affect the reading.

However, as can be demonstrated by calculations and from practical experience, this is responsible for only a very minor portion of the total temperature error, and is therefore usually disregarded.

In view of the fact that temperature errors may, in many cases, reach considerable dimensions, designs of a number of instruments for use within a wide temperature range provide special means of compensation. This will be discussed in a separate Section below.

In electric instruments, temperature errors may also develop due to changes in electric resistances, intensities of magnetic fields, etc.

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5. Errors due to Unbalance in Various Parts. When the position of the instrument is changed (i.e., when it is inclined in various directions), or when the point at which an instrument is installed is subjected to accelerations (i.e., when an aircraft is overloaded), the instrument will show a change in reading. This is due to the fact that the parts are not all in balance, and that some are generally not subject to balancing. The latter is the case when balancing would render the mechanism significantly more complicated, while a lack of compensation of particular parts does not produce excessive deviations in instrument readings.

Reduction in such error may be greatly facilitated by balancing all parts subject to more rapid motion than the rest. This derives from the fact that, in order to reduce a force to any given point (and therefore to a specific point of the pickup) this force must be multiplied by the corresponding velocity ratio. In this connection, even the slightest unbalance of the pointer may result in significant errors if the position of the instrument is changed. Unbalance of any part of the pickup may affect the instrument reading, even if the given parts are of considerable weight, since the displacement of the pickup is usually small.

Parts of the transmitting mechanism are more readily balanced, since they constitute simple levers. To balance the pickup as a whole is usually quite complicated. In this connection, the individual parts of many instruments are not balanced completely, and some are not balanced at all. For example, in centrifugal tachometers, the sleeve sliding along the tachometer shaft is not subjected to balancing. In certain diaphragm instruments, the pressure-measuring chamber itself is not balanced. As a result, inclination and acceleration have some effect on their readings.

Under laboratory conditions it is difficult to reproduce the effect of accelerations so that testing in this category is usually limited to checking the readings at various angles to the perpendicular, since this provides an adequate characterization of the balance of the mechanism. Specific allowances are made for tests

of this type in the instrument specifications.

6. Parallax Errors. Parallax is the name given to the phenomenon occurring when the readings of an instrument are taken from one side, a clearance being seen between the pointer and the dial calibrations.

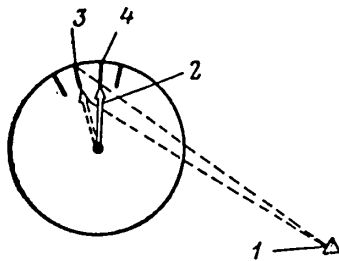


Fig.1 - Error in Reading from
the Parallax

Obviously, if the (Fig.1) eye is at the point (1), it will see the graduation (3) below the end of the pointer (2), although in reality it is graduation (4) that is below the pointer.

Fundamentally, the degree of parallax depends on the distance from pointer to dial face, so that this error may be classified with faults of design.

Parallax is partly dependent on the mounting of the instrument, since the degree of parallax is also governed by the angle at which the instrument must be read.

In assembling instruments, the pointer must be bent toward the dial face so as to leave the smallest possible clearance between them. Of course, at no reading must the pointer ever touch the dial.

Certain electric instruments present certain additional typical error sources, such as insufficient voltage in the feed circuit, etc.

Section 3. Methods of Calculating Particular Errors in Static Measurements

In instrument design, there are certain general methods that may be used for approximate computation of certain types of the error listed above. Specifically, there are general methods for calculating errors of friction, thermal error due to the instrument, and error due to unbalance of parts.

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Calculation of Frictional Error

To calculate errors due to friction, all moments and forces of friction in the mechanism must be reduced to a single point or axis, this being the pickup element of the instrument. In certain cases, this reduction must be performed for several instrument readings since, at different readings, the tension of the hairspring differs.

We know that the phrase "the total (or over-all) force (or moment) reduced to a given point (or axis) of a mechanism" refers to the equivalent force applied to this point; the simple work performed in simple displacement of the mechanism, is the sum of the simple work performed by all the forces and moments acting on various points in the mechanism, at the same displacement of the mechanism.

In accordance with the foregoing we have

$$\left. \begin{aligned} \text{or} \quad F_r dx &= \sum_{i=1}^{l=k} (F_i dy_i) + \sum_{n=1}^{n=m} (M_n da_n) \\ M_r d\varphi &= \sum_{i=1}^{l=k} (F_i dy_i) + \sum_{n=1}^{n=m} (M_n da_n), \end{aligned} \right\} \quad (1.1)$$

where F_r is the reduced force;

M_r is the reduced moment;

dx is the simple displacement of the point (in the direction in which the force is acting) to which all forces and moments are reduced ($d\varphi$ is the simple rotation of the axis to which the forces and moments are reduced);

F_i is a force acting on the mechanism;

M_n is a moment acting on the mechanism;

dy_i is the simple displacement of the point at which the force F_i is applied in the direction in which the force is acting; da_n is the simple rotation of the axis of the moment M_n , corresponding to the displacement dx of the

point (or the rotation $d\varphi$ of the axis) to which the forces and moments are reduced.

From eq.(1.1) we have

$$\left. \begin{aligned} F_{np} &= \sum_{i=1}^{l=k} \left(F_i \frac{dy_i}{dx} \right) + \sum_{n=1}^{n=m} \left(M_n \frac{d\alpha_n}{dx} \right); \\ M_{np} &= \sum_{i=1}^{l=k} \left(F_i \frac{dy_i}{d\varphi} \right) + \sum_{n=1}^{n=m} \left(M_n \frac{d\alpha_n}{d\varphi} \right). \end{aligned} \right\} \quad (1.2)$$

After all the forces and moments of friction in the mechanism have been reduced to the pickup, all that remains to be determined is the extent of change of the magnitude of the factor being measured, to compensate for the force of friction (i.e., to overcome that force). If, for example, the given moment of measurement is based on the principle of pressure measurement with some type of diaphragm, the frictional error, expressed in units of pressure, will be

$$\Delta p = \pm \frac{F_{fr}}{S_{eff}},$$

where Δp is the additional pressure needed to overcome friction;

F_{fr} is the reduced force of friction;

S_{eff} is the effective area of the diaphragm.

By "the effective area of the diaphragm" is meant the area of the equivalent piston, i.e., of a piston which, given a pressure difference p between its two sides, is required to maintain in equilibrium a concentrated force P of the same magnitude, that has to be applied to the center of the membrane to bring it into equilibrium when a differential pressure p is applied thereto.

The effective area of a diaphragm will be discussed in slightly more detail below, in the Section on membrane calculations.

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Let us examine a case in which the pressure p to be measured by the membrane is only an intermediate factor, functionally dependent on the quantity determined by the instrument. Then, in determining the degree of frictional error expressed in pressure units, it is possible, by using the given functional ratio, to determine the approximate error, now expressed in units of specific size, from the equation:

$$\Delta N \approx \Delta p \left(\frac{dN}{dp} \right)$$

or

(1.3)

$$\Delta N \approx \frac{\Delta p}{\left(\frac{dp}{dN} \right)},$$

where N is the quantity measured by the instrument. The derivatives $\frac{dN}{dp}$ or $\frac{dp}{dN}$ are taken as the value of the measured quantity under discussion (the one used is that more convenient in the given instance).

Equations (1.3) may also be used if the analytical relationship between the quantities N and p is not given and is deduced only graphically, i.e., if the curve for this relationship is given. It is obvious that, in this case, the magnitude of the derivative $\frac{dp}{dN}$ is determined by the slope of the tangent to the given curve, at a point corresponding to the value of N under examination.

At this point let us derive certain examples of the application of the propositions given above.

According to the approximate formula for the index of velocity (in the given case, the use of this approximate formula is entirely in order, since the entire calculation of stagnation is approximate), we have

$$p = \frac{\gamma v^2}{2g},$$

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where v is the speed of an aircraft;

γ is the density of the air by weight;

g is the acceleration of gravity. In accordance with the theorem of finite differences

$$\Delta p = \frac{\gamma v_{cp}}{g} \Delta_1 v,$$

where v_{mean} is some mean velocity in the interval $\Delta_1 v$ under examination. Since this range is very narrow, we may write, in first approximation,

$$\Delta p \approx \frac{\gamma v}{g} \Delta_1 v,$$

from which we derive

$$\Delta_1 v \approx \frac{g}{\gamma} \frac{\Delta p}{v}.$$

The magnitude of $\Delta_1 v$ gives us the extent to which the velocity v has to be increased (or decreased, if Δp is negative), to bring the forces of friction in the instrument into equilibrium, i.e., to enable the instrument to indicate the velocity v . However, since we are not actually making the indicated addition $\Delta_1 v$ to the velocity v , there will be distortions in the instrument reading to the reciprocal of the quantity $\Delta_1 v$, or, in other words, the instrument error Δv will be

$$\Delta v \approx - \frac{g}{\gamma} \frac{\Delta p}{v},$$

Since

$$\Delta p = \pm \frac{F_{TP}}{S_{\Phi}},$$

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it follows that

$$\Delta v \approx \mp \frac{g}{\gamma} \frac{F_{fp}}{S_{\phi} v}.$$

Thus, the error due to friction in an airspeed indicator, expressed in units of velocity, is directly proportional to the error expressed in pressure but inversely proportional to the velocity being measured.

In view of the foregoing, the hairspring in speed indicators is always so mounted that its tension will be lowest at the minimum value the instrument can measure. As demonstrated below, the tension in a mechanism resulting from the spring backlash results in some rise in the forces of friction between the individual parts of the mechanism.

Despite this fact, the friction error in speed indicators is considerably greater at low speeds than at high speeds since the magnitude F_{fr} of the reduced force of friction in the mechanism and the effective area S_{eff} vary considerably less than the magnitude of the velocity v being measured.

Under conditions in which the pressure in an instrument is not measured by means of a diaphragm or sylphon but by means, for example, of a Bourdon tube, for which it is always difficult to define the effective area, it is only necessary to determine the required pressure by the deflection at the end of a Bourdon tube, to reproduce the applied friction acting in the direction of the link connecting the end of the tube to the rest of the mechanism of the instrument. This may be accomplished in the following manner, for example: In the works of V.I. Feodos'yev, formulas are derived for the flexure of a Bourdon tube element bounded by two cross sections, both under the effect of a differential pressure p and under the effect of an external flexure moment M . For a thin-walled Bourdon tube of elliptical cross section, we have the following equation:

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$$\left(\frac{\Delta(d\theta)}{d\theta} \right)_p = p \frac{1-\mu^2}{E} \frac{\rho^2}{bh} \left(1 - \frac{b^2}{a^2} \right) \frac{\alpha}{\beta + \lambda^2},$$

where

Here $d\theta$ is the angle between the two cross sections of the tube, and $\Delta(d\theta)$ is the variation in this angle under the differential pressure p ;
 E is the modulus of elasticity of the material of which the tube is made;
 μ is Poisson's ratio;
 ρ is the radius of curvature of the central layer of the tube;
 h is the wall thickness;
 a and b are the cross-sectional radii of the tube (in Bourdon tubes, the cross section is oval or elliptical);
 α and β are dimensionless coefficients, whose magnitude is a function of the $\frac{b}{a}$ ratio (V.I. Feodos'yev offers tabular compilations).

The formula for deformation under the influence of an external moment is

$$\left(\frac{\Delta(d\theta)}{d\theta} \right)_M = M \frac{1-\mu^2}{E} \frac{\rho}{k_e J_a},$$

where J_a is the moment of inertia of the cross-sectional area of the tube relative to its diameter $2a$;

k_e is a correction for the phenomenon noted as far back as by Karman in hollow tubes under flexure, and described in courses on the theory of elasticity. According to the derivation by V.I. Feodos'yev, this correction k is equal to the following in an elliptical pipe cross section

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$$k_s = 1 - \frac{\zeta}{\xi + \lambda^2},$$

where ζ and ξ are dimensionless factors, whose magnitude is a function of the $\frac{b}{a}$ ratio and is provided by V.I. Feodos'yev in tabular form.

To determine the displacement of the end of a Bourdon tube, due to the above-discussed deformations, let us take as the origin of coordinates the free end O of a tube, in some given position (Fig.2), while for the Ox and Oy axes coordinates, any mutually-perpendicular straight lines passing through the point O may be used. Then, on flexure of an element N through an angle $\Delta(d\theta)$, in a tube of $\rho d\theta$ length (measured along its center line), the free end of the spring undergoes displacements df_x and df_y along the axes Ox and Oy, to a magnitude of

$$\begin{aligned} df_x &= -y \Delta(d\theta); \\ df_y &= x \Delta(d\theta), \end{aligned}$$

in which x and y are the coordinates of the point N.

From this we determine that, on flexure of all elements of the tube, the free end of the tube will be displaced as follows:

$$\begin{aligned} f_x &= -\int y \Delta(d\theta); \\ f_y &= \int x \Delta(d\theta), \end{aligned}$$

the integral being determined within the limits along the entire length of the tube.

If the direction of the axes of the coordinates is so selected that the axis Oy is the continuation of a link connecting the O end of the tube with the rest of the mechanism or, in other words, so that the external load Q acts in the direction of the Oy axis, the external moment M will be

$$M = Qx$$

Therefore, the deformations $(f_x)_p$ and $(f_y)_p$ resulting from the action of the

external force Q will be

$$(f_x)_Q = -Q \frac{1-\mu^2}{E} \frac{\rho}{k_c J_a} \int xy d\theta;$$

$$(f_y)_Q = Q \frac{1-\mu^2}{E} \frac{\rho}{k_c J_a} \int x^2 d\theta;$$

while the deformations $(f_x)_p$ and $(f_y)_p$ resulting from the action of the differential pressure p are equal to

$$(f_x)_p = -p \frac{1-\mu^2}{E} \frac{\rho}{bh} \left(1 - \frac{b^2}{a^2}\right) \frac{\alpha}{\beta + \lambda^2} \int y d\theta;$$

$$(f_y)_p = p \frac{1-\mu^2}{E} \frac{\rho}{bh} \left(1 - \frac{b^2}{a^2}\right) \frac{\alpha}{\beta + \lambda^2} \int x d\theta.$$

The error due to friction can only be approximately determined since the force of friction is itself quite inconstant and dependent for its magnitude on a number

of circumstances. Therefore, in calculating the magnitude of the error, the usual procedure is to make only a comparison of the deflection due to an external force Q with that due to pressure p only in the direction Oy of the link, connecting the end of the tube to the rest of the mechanism.

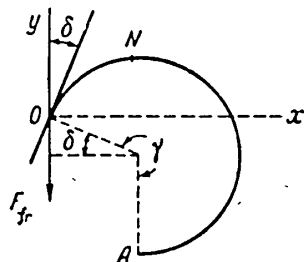


Fig.2 - System of Coordinates Used in Calculating the Magnitude and Direction of Deflection of a Bourdon Tube

Note: In manometers, $(f_x)_p$ is usually close to zero, since these are so constructed that

the direction of the link, connecting the end of the tube to the rest of the

mechanism, coincides with the direction of motion of the tube end on deflection, subject to the action of differential pressures.

Thus, the frictional error Δp_{fr} , expressed in pressure units, may be approximately calculated in instruments with Bourdon tubes by means of the following equation:

$$\Delta p_{fr} \frac{\rho}{bh} \left(1 - \frac{b^2}{a^2} \right) \frac{\alpha}{\beta + \lambda^2} \int x d\theta = \pm F_{fr} \frac{1}{k_e J_a} \int x^2 d\theta. \quad (1.4)$$

If an ordinary Bourdon tube, bent into an arc of a circle, is involved, it is a simple matter to express the magnitude x within the expressions being integrated in both the integrals above, as a function of the angle θ . The magnitude of y may be determined similarly if, for some reason, it is also necessary to determine the degree of deflection of the tube in the Ox direction. At the same time, obviously, the magnitude of the limits of integration are also determined. Figure 2 indicates that these magnitudes can be expressed as follows:

$$\begin{aligned} x &= \rho (\cos \delta - \cos \theta); \\ y &= \rho (\sin \theta - \sin \delta), \end{aligned}$$

the limits of integration of all the integrals being

$$\text{from } \delta \text{ to } \gamma + \delta,$$

where δ is the angle between the direction of the Oy axis and a tangent at the point O to the circumferences of the center layer of the tube,
 γ is the angle formed by the arc of the tube.

If, in this indirect method of measurement, the function of the instrument consists in measuring some force P by means of a spring [this force P being related to the factor N under measurement, by an equation having the form $N = f(P)$ or $P = \phi(N)$], as occurs for example in the centrifugal tachometer, it is obvious that the frictional forces in the mechanism must be referred to the point of action of

the force of the spring. In this situation, we have equations similar to those in the system (1.3) for determining the error ΔN :

$$\left. \begin{aligned} \Delta N &\approx \pm F_{fr} \frac{dN}{dP}, \\ \Delta N &\approx \pm \frac{F_{fr}}{\frac{dP}{dN}}. \end{aligned} \right\} \quad (1.5)$$

As another example, let us consider a thermograph using a bimetal plate as its pickup. Knowing the force of friction at work in the mechanism (including, of course, that of the stylus on the paper), it is necessary to determine the deformation of the bimetal plate under the influence of this force at the point of action of the force. Having determined the deformation that may result from the application of a force equal to the reduced force of friction, it becomes necessary to calculate the necessary change in temperature in order to obtain a reverse deformation of the bimetal plate at the same point and of the same magnitude. The quantity thus obtained is the error of friction of the given thermograph.

In other words, if we have the following formula for thermal deformation of the bimetal plate

$$\Delta f_t = \Phi(\Delta t),$$

where $\Phi(\Delta t)$ is a particular function of a parameter of the bimetal plate and of the change in temperature Δt , and if the formula for deflection of the plate Δf_p , on mechanical loading, is

$$\Delta f_p = \theta(P),$$

where $\theta(P)$ is a particular function of the parameters of the bimetal plate and load P , then the error under examination is found from the equation

$$\Phi(\Delta t_{fr}) = \theta(F_{fr}),$$

where F_{fr} is the total friction of the mechanism.

Calculation of Error Due to Unbalance of Parts

In calculating errors due to unbalance of the parts of a mechanism, we may proceed in the same fashion as in calculating errors due to friction. In this situation, all the formulas derived above for calculating this error are applicable, but the reduced force of friction is replaced by the force of unbalance of the parts of the mechanism, reduced to the elastic element.

General Methods of Calculating the Temperature Error of Instruments

As already indicated, the major source of thermal error in instruments with elastic pickups is a variation in the modulus of elasticity of the material of which the elastic element is composed.

In determining the resultant changes in instrument readings or the magnitude of the resultant change in deformation of the elastic element, we will use the following example for the sake of simplicity.

Although, in point of fact, any change in temperature is accompanied only by a deformation of the elastic element as a result of the change in the modulus of elasticity E , while the load P or p remains constant, we intentionally subdivide the process into two stages. First, by changing the temperature, we consciously vary the load in such a fashion that the deflection remains constant, after which we remove this spurious extra load. This latter step is equivalent to applying a new supplementary load of equal magnitude but opposite sign. This permits determining the manner in which the instrument readings change.

Now let us first determine to what degree the load on the elastic element may be changed with any change in temperature, without resulting in a change in deflection.

For all types of springs in which the stress-strain ratio is linear with STAT concentrated load, the formulas determining the relationship between the load and the corresponding deformation have the following general appearance:

$$P = EFx,$$

or

$$P = G\Phi x,$$

where P is the load;

x is the corresponding deflection;

E is the modulus of elasticity of the spring material;

G is the shear modulus.

The quantities F and Φ are functions of the spring dimensions. These functions are generally of the same order of dimensions as the spring itself: Thus, in a flat rectangular spring we have $F = \frac{bh^3}{4l^3}$, where b is the width of the spring, h is its thickness, and l its length.

In the case of a distributed load p (sometimes termed a hydrostatic load), expressed in units of force per unit area (i.e., kg/cm²), we would have $p = E\varphi$, where φ is some function of the deflection of an elastic element (diaphragm or Bourdon tube) and of its dimensions. In this situation, given a linear relationship between pressure and deflection (as in Bourdon tubes), we have $\varphi = \psi x$, where x is the deflection of some given point of the elastic element, and ψ is a function of the dimensions of the elastic element. In this case, the function ψ generally is of the first negative order of dimensions of the elastic element.

In a case in which the relationship between the pressure p and the corresponding deflection is not linear, as for example in a membrane, or in other words if the elasticity of the diaphragm is not constant, φ is a more complex function of the deflection and the dimensions of the elastic element. In this case, the function φ is generally dimensionless.

Any change in temperature is accompanied by a change in the modulus of elasticity (or the shear modulus) of the spring material. In the majority of metals, the modulus of elasticity increases as the temperature decreases and declines as it

increases.

This law may be expressed, in first approximation, by the following equation:

$$E_t = E_0(1 - \beta \Delta t),$$

where E_t is the modulus of elasticity after the temperature change;

E_0 is the modulus of elasticity at the initial temperature;

β is a factor called the temperature factor of the modulus of elasticity;

Δt is the change in temperature.

The magnitude of the temperature coefficient for various metals is derived in handbooks, although this factor is usually given for 0°C as the initial temperature, at which the modulus of elasticity is E_0 . The modulus of shear G varies with any variation in the modulus of elasticity E .

Thus, if the temperature changes, a slight change in the load is required in order to obtain the former deflection x .

If, at an initial temperature, $P = E_0 F_x$ (or $P_0 = G \psi x$) in the case of a concentrated load and $p_0 = E_0 \psi x$ in the case of a distributed load and at a linear relationship between pressure and deflection, then, after the temperature has changed, we will have

$$P_t = E_0(1 - \beta \Delta t) F(1 + \alpha \Delta t) x$$

or

$$P_t = G(1 - \beta \Delta t) \Phi(1 + \alpha \Delta t) x$$

and

$$p_t = E_0(1 - \beta \Delta t) \frac{\psi}{1 + \alpha \Delta t} x,$$

where α is the thermal coefficient of elongation of the spring material.

From this, and neglecting small quantities of the second order, i.e., terms in which the product of $\beta \Delta t$ and $\alpha \Delta t$ appears, we have

$$\left. \begin{aligned} \Delta P &\approx -E_0 F_x (\beta - \alpha) \Delta t = -P_0 (\beta - \alpha) \Delta t; \\ \Delta p &\approx -E_0 \psi x (\beta + \alpha) \Delta t = -p_0 (\beta + \alpha) \Delta t. \end{aligned} \right\} \quad (1.6) \text{ STAT}$$

In the case of a nonlinear relationship between the pressure p and the deflection x , we obtain the following, in a similar fashion:

$$\Delta p = -E_0 \varphi [\beta + f(\alpha)] \Delta t = -p_0 [\beta + f(\alpha)] \Delta t, \quad (1.7)$$

where $f(\alpha)$ is some more complex function of α .

The coefficients of thermal elongation of metals α are considerably smaller than the heat coefficients β of the modulus of elasticity of the same metals, and therefore they are usually ignored in practice. This is the more permissible since, in the calculation, it is often impossible to consider such factors as residual tension in the spring or diaphragm manufacture. Nor can one adhere completely to the specified dimensions of springs. In addition, it is impossible to allow for inhomogeneities in the material or other factors that may result in greater distortions of the calculated results. With this simplification, we obtain simpler expressions

$$\left. \begin{aligned} \Delta P &= -P_0 \beta \Delta t; \\ \Delta p &= -p_0 \beta \Delta t. \end{aligned} \right\} \quad (1.8)$$

It must be considered that, in examining the thermal errors in an instrument, we really do not actually face the changes in the relationship of forces referred to above and acting on the pickup. When the temperature changes, the force P or pressure p does not change, so that the change in the instrument readings will have to be determined. It is obvious that a change ΔP_t (or Δp_t) in the instrument reading, expressed in units of force P (or of pressure p), may be considered opposite but equal in absolute magnitude to the quantity ΔP (or Δp) derived from eq.(1.8). Of course, if, in order to obtain the former deflection of the elastic element, a lower pressure proves to be required, then, given no change in pressure, the deflection will increase and the instrument will present a higher reading. Thus,

$$\left. \begin{aligned} \Delta P_t &= P_0 \beta \Delta t; \\ \Delta p_t &= p_0 \beta \Delta t. \end{aligned} \right\} \quad (1.9) \text{ STAT}$$

Knowing the thermal instrument error, expressed in units of force P or of pressure p , it is easy to determine the error being sought expressed in units of

the magnitude being measured, as we do know the functional relationship between the magnitude being measured and the force P or pressure p .

It is quite obvious that here, too, formulas analogous to eq.(1.3) are entirely applicable and that, for theoretical calculations of the thermal instrument error ΔN_t , expressed in units of the magnitude being measured N , the following equations can be used

$$\left. \begin{aligned} \Delta N_t &\approx \Delta P_t \frac{dN}{dP} = \frac{\Delta P_t}{\frac{dP}{dN}} \\ \text{and} \quad \Delta N_t &\approx \Delta p_t \frac{dN}{dp} = \frac{\Delta p_t}{\frac{dp}{dN}} \end{aligned} \right\} \quad (1.10)$$

However, if it is required to determine the thermal error Δx_t , expressed in units of deflection x of the pickup (i.e., in the temperature calculation of instruments), then the terms $\frac{dP}{dx}$ or $\frac{dp}{dx}$ in eqs.(1.3) must be substituted for $\frac{dP}{dN}$ or $\frac{dp}{dN}$

$$\left. \begin{aligned} \Delta x_t &= \Delta P_t \frac{dx}{dP} = \frac{\Delta P_t}{\frac{dP}{dx}} \\ \Delta x_t &= \Delta p_t \frac{dx}{dp} = \frac{\Delta p_t}{\frac{dp}{dx}} \end{aligned} \right\} \quad (1.11)$$

Strictly speaking, in eqs.(1.11) the terms $\frac{dP}{dx}$ or $\frac{dp}{dx}$ are not exactly the same as at the temperature under analysis, since the elasticity of the spring varies with any change in temperature; therefore, in the expressions derived above, $\frac{dP}{dx}$ and $\frac{dp}{dx}$ must be taken as equal not to the calculated quantities $\left(\frac{dP}{dx}\right)_0$ and $\left(\frac{dp}{dx}\right)_0$, STAT

corresponding to the initial temperature under analysis, but to the quantities

and

$$\left(\frac{dP}{dx}\right)_t = \left(\frac{dP}{dx}\right)_0 (1 - \beta \Delta t)$$

$$\left(\frac{dp}{dx}\right)_t = \left(\frac{dp}{dx}\right)_0 (1 - \beta \Delta t).$$

Therefore eqs.(1.11) may be written in the form

and

$$\left. \begin{aligned} \Delta x_t &\approx \frac{\Delta P_t}{\left(\frac{dP}{dx}\right)_0 (1 - \beta \Delta t)} \\ \Delta x_t &\approx \frac{\Delta p_t}{\left(\frac{dp}{dx}\right)_0 (1 - \beta \Delta t)} \end{aligned} \right\} \quad (1.12)$$

However, in view of the fact that the term $\beta \Delta t$ in parentheses is usually much smaller than the other terms, it is very often neglected.

If we proceed in the same fashion as in determining the frictional error of the speed indicator, on the basis of the expression

$$\Delta v \approx \frac{g}{\gamma} \frac{\Delta p}{v}$$

we obtain

$$\Delta v \approx \frac{1}{2} v \beta \Delta t.$$

Obviously, the temperature error of the speed indicator increases with the velocity, unlike its error of friction.

It should be noted that, in instruments in which the pickup is a chamber consisting of two diaphragms soldered to each other around the edges, the thermal error does not always strictly follow the relationship expressed by eq.(1.9). In this type of double diaphragm, the thermal error is often zero not at a zero

pressure p , as would follow from eq.(1.9), but at some positive or negative pressure (although usually low). Then this temperature error can be expressed approximately by the following formula:

$$\Delta p = (b - p_0 \beta) \Delta t,$$

where b is some constant for the given diaphragm chamber.

In this case, the temperature error is already present at zero pressure, meaning that the diaphragms acquire some degree of deflection in the absence of pressure and solely from the change in temperature.

This may be explained by residual tensions in the diaphragm material after its manufacture and in part by the fact that the chamber components (the diaphragms proper, rigid centers soldered to the diaphragms, the metal used in soldering) are made of various materials which may differ in their coefficients of thermal expansion.

The magnitude b varies greatly in various diaphragm chambers, and it is therefore theoretically incapable of being determined in advance. As a result, an empirical determination of chambers already manufactured must be made in order to interpret certain results of measurement, and also in deciding on the temperature compensation for the instrument. However, if a preliminary approximate determination of the thermal error is required prior to manufacture of the diaphragm chamber, eq.(1.9) is to be used.

Section 4. General Information on Instrument Setting

As previously indicated, in the series manufacture of instruments, individual parts of their mechanisms may not be identical in different instruments of the same lot. This pertains specifically to the pickups. Even if identical springs of the simplest type are used, the total elasticity and characteristics sometimes differ noticeably. As far as more complex pickups are concerned, such as diaphragms for pressure measurement, they often differ notably both in total elasticity and in

characteristics.* Similar inconstancies can be observed in other types of pickups.

This situation means that, in order to obtain more or less identical scales for identical instruments, it is impossible to use levers of completely identical dimensions in the transmitting elements and to place them in completely identical initial positions. Therefore, in instrument design it is necessary to provide for adequately broad possibilities of variation both in the total transmission ratio of the mechanism and in its characteristic, i.e., in the nature of the interdependence of the displacements of the first and last elements in the mechanism. In so doing, advantage is taken of the circumstance that the majority of elementary transmission connections used in instruments - to be described in a special Chapter - themselves have inconstant transmission ratios.

In the present Section, for purposes of illustration, the discussion will be limited to certain general considerations involved in the setting of instruments.

The major object in instrument adjustment is a reduction of the errors of scale. This is partially attainable by merely varying the relative position of various parts of the pickup. However, this type of regulation varies from one type of instrument to the next, and examples of its use will be cited in our study of specific instruments.

Usually, however, the basic adjustment of instruments consists in changing both the dimensions and the relative positions of individual parts of the transmitting mechanism. For this type of adjustment, a number of generally applicable methods and considerations can be given.

Errors of scale in instruments result primarily from the two following causes:

1. The pointer is improperly seated on its axis. In this case, errors will be of the same angular magnitude in all readings of the instrument. Expressed in STAT

* By characteristic we mean the functional ratio of the load to the deflection of the elastic element.

units of the quantity to be measured, the errors may not be identical for various portions of the scale, since not all instruments have equidistant spacing. In the present case, adjustment is easy. All that is necessary is to seat the pointer properly.

In certain instruments (for example, certain speed indicators), the pointer still does not come to rest at the zero point on the scale when the velocity to be measured is zero, even after the pointer has been relocated. However, the mechanisms of most of these instruments provide for a stop, whose position may be changed in such fashion as again to cause the pointer to stop at zero if the measured quantity is equal to zero.

Of course, exceptions may exist here as well. For example, in altimeters, the very principle of operation means that even at ground level, or zero altitude, the pointer does not stop at a fixed point, since it fluctuates with the atmospheric pressure. Therefore, an altimeter must not have the type of stop referred to above, and an adjustment of this type cannot be used.

2. The transmitting mechanism fails to transfer the displacement of the pickup to the pointer of the instrument, at the ratio required for the instrument to give correct readings.

Cases may arise in which the transmission ratio of the mechanism is excessively high at all readings. In this situation, the instrument will yield exaggerated readings all along the scale, and the error will increase with higher readings.

If the transmission ratio is too small throughout, the instrument will yield reduced readings all along the scale, although the absolute magnitude of the error will also increase with an increase in the instrument readings.

The nature of the error $\Delta\alpha$ in instruments is shown in angular units α in terms of their readings (also in angular units) in Fig.3, where the solid line represents excessive transmission ratios and the broken line gives excessively low transmission ratios. Figure 4 shows a combination of this type of error with that discussed

above when, in addition, the pointer is improperly seated.

Cases are not uncommon in which the transmission ratio is too small along one

portion of the scale and too large elsewhere. In this situation, the instrument error may first increase in some direction (e.g., on the positive side) and then start to drop, often even changing in sign (Fig.5).

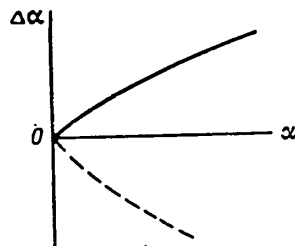


Fig.3 - The Nature of Error When the Transmission Ratio of the Mechanism is Too Large or Too Small Throughout the Range of Measurable Values

In all types of error referred to above, it is necessary to adjust so as to attain a situation in which the required transmission ratio is produced by the transmission mechanism all along

the scale.

In terms of geometry and also on the basis of the equations derived below in the Section on the fundamental types of transmission, the following basic rules for adjustment of transmission mechanisms can be given:

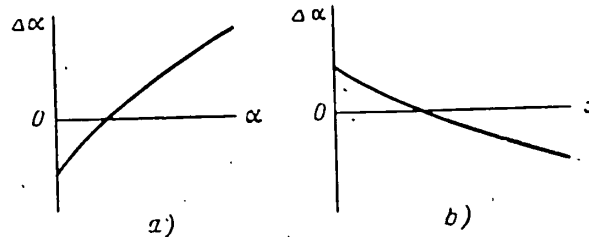


Fig.4 - The Nature of Error When the Transmission Ratio of the Mechanism is Too Large or Too Small and if an Error is Already Present at Zero Reading.

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1. The larger the drive arms and the smaller the driven arms of levers in the

transmitting mechanisms, the greater will be the rate of displacement of the pointer.*

2. The smaller the drive arms of the levers in the transmitting mechanism and the larger the driven arms, the smaller will be the transmission ratio of the instrument.

These two rules hold for all types of transmission connections. Within the meaning of these rules, rockers are also regarded as lever arms.

As far as the angles between the parts of the mechanisms are concerned, no general rule such as the foregoing is derivable in this connection. Moreover,

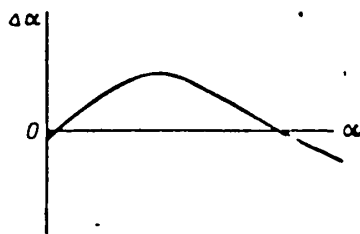


Fig.5 - Nature of Error When the Transmission Ratio is Excessive in One Area of Operation of the Instrument, and Too Small Elsewhere

entirely contradictory rules may be derived from various types of transmitting connections. Therefore, the various types of connection have to be examined separately. However, certain general observations can be derived in the given case.

Most frequently, a general increase or decrease (i.e., one occurring along the entire scale) in the transmission ratio of the mechanism is obtained by increasing or decreasing the dimensions of the lever arms of the transmitting mechanism. If the instrument readings are low along the entire scale, one must either increase the drive arms or reduce the driven lever arms. For this purpose, the transmitting mechanism of most instruments is made in such a fashion as to make it possible to change the lever-arm dimensions.

*A definition of the concept of the "drive" or "driven" arm of a transmitting mechanism lever is given in Section 1 of Chapter IX.

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However, if the instrument errors are such that it is necessary to increase the transmission ratio for one part of the scale and to reduce it at another, the instrument is usually adjusted by seeking to determine the principle of change in the angles between the transmitting mechanism, as it is displaced. Obviously, these angles undergo constant change as the transmission mechanism is displaced.

Let us assume that, at some segment of the scale, the transmission ratio is very small relative to the same ratio along other segments. Consequently, the transmission mechanism must be so adjusted that, specifically along this segment of the scale, the angles between the transmission parts yield the largest movement of the pointer. The Chapter on standard transmission connections will show the manner in which the magnitude of these angles affects the magnitude of the transmission in a given transmission connection. At this point, only the qualitative effect of the angles between parts in certain specific types of mechanisms will be discussed.

In examining transmission connections, it is not difficult to prove that all connections involving rotating joints (chain connections, for example) are subject to the following rules:

1. The less the angle between the drive arm of the lever and the link connected thereto by a rotating joint differs from a right angle and, conversely, the more the angle between the driven arm of the lever and the link connected thereto differs from 90° (regardless of whether it is more obtuse or more acute), the greater will be the transmission ratio of the mechanism.

2. The more the angle between the drive arm of the lever and the link connected thereto by a rotating joint differs from a right angle, and the less the angle between the driven arm of the lever and the link connected thereto differs from 90° , the lower will be the transmission ratio of the mechanism.

All changes in the size of the angles between adjacent parts of the transmission mechanism have a particularly pronounced effect on the transmission ratio of the mechanism at very acute or very obtuse angles (i.e., at angles close to 0°

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or 180°) between the individual parts. At such angles, an insignificant change in the magnitude of the angle may have a very pronounced effect on the rate of motion of the indicator. However, when the angles between the parts of a transmission mechanism are close to a right angle, any fluctuation in the magnitude of these angles is very weakly reflected in the transmission ratio.

These considerations must be borne in mind whenever the transmission includes a number of angles between individual parts and when a change in one of these angles in the direction desired for regulation causes another to change in a direction not desired for this regulation.

Let us take the case depicted in Fig.6 as a typical example.

Let A be the pickup part of the mechanism and the levers AC and DF be parts of the transmission, rotating on the axes B and E and both hinged to the lever CD. As the instrument readings increase, let the point A move in the direction shown on the drawing by an arrow. Let us assume that, with a transmission so arranged, it is necessary to increase the transmission ratio in the range of maximum readings, without increasing it for other areas (i.e., the instrument reads too low at high readings). With this object, the instrument must be assembled in such a fashion as to cause the transmission to operate at more obtuse angles between the thrust CD and the lever DF (since DE is the driven arm of the lever). Naturally, this will cause the angles between the drive arm BC and the link CD to be more acute, but this will have a less marked effect on the transmission ratio.

Assembly of the mechanism in this fashion may to some degree increase the transmission ratio at all readings of the instrument, while this increase will be greater at larger readings (where the CDE angle is more obtuse) than at smaller readings (where the angle is less obtuse).

The general increase in the transmission ratio along the entire scale, resulting from the foregoing, may be compensated by a change in the length of some lever arm in the mechanism. To make this kind of adjustment possible, the pickup portion

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of many instruments is capable of being displaced relative to the transmission mechanism. This approach is not uncommon in adjusting individual instruments, such as recording altimeters.

In individual cases, other considerations may influence the decision as to which angle change will have the greater effect on the motion of the pointer. For example, if the arms of two levers connected by a link differ in length, the angles made by the link with the shorter arm will change more rapidly than that with the longer arm. In general, in transmissions in which several changing angles are a

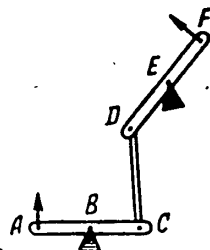


Fig.6 - Position of Mechanism when a Rotation of the Lever BA Causes the Transmission Mechanism to Undergo a Greater Change due to Variations in the Angle CDE than due to Variations in the Angle BCD

factor, it is always necessary to determine, by reasoning, the change in angle that will have the greater effect on the corrections by the instrument.

Sliding connections such as those shown in Figs.69 and 70, which may be termed "sinusoidal" sliding connections, are obviously subject to the same rules as rotating connections, but in this case it is not the angle between lever and link that has to be considered, but that between the lever OB and the direction of displacement of the plane on which the lever rests.

Sliding connections such as those shown in Fig.71, which may be denoted as "tangential", are obviously subject to other laws, as is clear from a comparison of eqs.(9.12) and (9.14).

It is obvious that the rules involved here are the converse of those to which sinusoidal connections are subject, which we had discussed for the case of hinged connections. In applying these rules to the present case, the angles between the

levers and links must be replaced by angles between the lever OB and the direction of motion of the point C, where the lever contacts the driving portion of the mechanism.

In the case of the dog transmission depicted in Fig.72, we have (as explained below) a combination of two tangentially sliding connections, in one of which the lever is of the driving type and in the other, of the driven type. Further, if the axes (1) and (2) lie in the same plane, then, in positions of the mechanism in which one of these tangential connections provides a gradual reduction in transmission ratio, the other tangential connection results in a gradual increase in the transmission ratio. Therefore, if the terms a and b [eqs.(9.16) and (9.17)] are equal, the transmission ratio of the given dog connection remains constant in all positions. However, if a and b are not equal, then the transmission ratio will be more strongly affected by the tangential connection in which the lever (or, to be exact, the distance from the axis of rotation to a straight line constituting the geometrical point of contact of the dogs) is shorter, since in this connection the angles will change more quickly. Thus, if $b < a$, the transmission ratio will be greatest when the transmission is in the center position and the dogs are in the plane containing the axes (1) and (2).

It is clear from simple geometrical considerations that the bend in the dog (3), shown in Fig.73, leads to a reduction, and the bend shown in Fig.74 leads to an increase in the transmission ratio as soon as the mechanism deviates from the center position.

If, in the simple rocker connection shown in Fig.76, the lever OB is the leading lever, the maximum transmission ratio will apparently be that existing when the points O, B, and O_1 are on a single straight line, i.e., when the transmission is in the center position.

Similar considerations result for other types of transmissions. Often they may be made on the basis of general geometrical considerations, without the deriva-

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tion of exact formulas for the given transmission ratio. But in many cases a knowledge and accurate comprehension of the analytical formulas of transmission connections is essential in adjustment. We will describe such formulas (for the most common types of connections) in a Section devoted to this subject.

Section 5. Sensitivity and Accuracy of Measuring Instruments

In examining errors of measuring instruments, it is necessary to pay attention to two concepts: the "sensitivity" and the "accuracy" of measuring instruments.

The sensitivity of a measuring instrument is the "ratio of the linear or angular displacement of the indicator to the change in the quantity being measured that causes this displacement".

It may be stated in passing that the "threshold" of sensitivity of a measuring instrument is the "smallest change in the value of the magnitude to be measured, capable of inducing the smallest change in indication by the measuring instrument".

The accuracy of a measuring instrument is the "degree of reliability of the results of measurement, evaluated by indicating the positive and negative limits (+) of the maximum error, or the same limits of one of the mean errors, as established by the theory of accidental errors (mean-square or more likely, mean-arithmetic)".

Proper selection of the sensitivity of an instrument for specific concrete conditions of measurement is a very significant aspect of the theory of measurements. The point is that the sensitivity of an instrument characterizes the accuracy of an instrument only in terms of a single category of errors, i.e., errors of indication.

We have already indicated that, in addition to this category of errors, measuring instruments are subject to intrinsic mechanical errors, and sometimes to errors of method. It is obvious that corresponding rational relationships exist between the reading errors pertaining fundamentally to the sensitivity of the instrument, and all the other errors. It makes no sense to effect an extreme reduction of one category of error while permitting rather large errors to continue to exist in other

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categories. There are cases in which this may actually be harmful. The latter may occur when a reduction in one of the errors causes an increase in another and further complicates the instrument, while an increase in sensitivity involves a possible overestimation by the observer of the reliability of the result obtained.

Let us cite one example of the need for maintaining corresponding ratios between the errors and in particular between the sensitivity of an instrument and the errors of friction.

In a number of cases, we may be able to assume, to a certain degree of accuracy, that the frictional error is proportional to the transmission ratio i . In recording instruments, where the greatest friction is at the end of the pen, this is virtually the case. True, in the remaining instruments we only have something similar to this, since the rate of displacement does not increase in all parts to the same degree as does the increase in the transmission ratio. It must be borne in mind, however, that additional parts may be needed if the transmission ratio is larger. Thus, in many cases we are in a position to write, in first approximation,

$$\frac{\Delta_{fr}}{i} \approx C$$

where Δ_{fr} is the frictional error;

i is the transmission ratio of the transmitting mechanism, i.e., the ratio of the rate of displacement of the indicator to the rate of displacement of that point in the pickup whose motion is transmitted to the indicator;

C is a constant.

The indicator error may be taken on the average to be inversely proportional to the scaling, i.e., inversely proportional to the transmission ratio i . Thus, we have

$$\Delta_{ind.} \approx \frac{k}{i},$$

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where Δ_{ind} is the error of indication,

k is some constant factor.

The total error of friction and indication is approximately equal to

$$\Delta = \Delta_{fr} + \Delta_{ind} = Ci + \frac{k}{i}$$

To clarify the question as to the value of i at which the total error will be minimal, let us take the derivative from Δ to i and equate it to zero. We then obtain

$$C - \frac{k}{i^2} = 0$$

and, multiplying this equation by i , we will have

$$Ci = \frac{k}{i}$$

or

$$\Delta_{fr} = \Delta_{ind} \quad (1.13)$$

In order to confirm whether the given ratio corresponds to a minimum or maximum value of Δ , let us take the second derivative of Δ along i . This is equal to

$$\frac{d^2\Delta}{di^2} = +2 \frac{k}{i^3}$$

Obviously, the second derivative is positive and therefore satisfaction of eq.(1.13) is a condition for obtaining a minimum value of Δ .

If the frictional error is not exactly proportional to the first power of the transmission ratio, the most favorable ratio of Δ_{fr} to Δ_{ind} will be somewhat different. In this case, however, the reasoning derived above provides some basis for determining approximately what must be the relationship between these two errors, STAT

in order for total error to be as small as possible, i.e., what must be the magnitude of the most desirable transmission ratio which ultimately determines the sensitivity of the instrument.

However, as we know, the sources of instrument error include a considerable number of errors whose magnitude is virtually independent of the transmission ratio. Often the total instrument error is many times larger than the total error due to sluggishness and inaccuracies of indication. In this case, it is not too important if the condition cited above as to minimum total error due to sluggishness and inaccuracy of indication is not fully adhered to, since the relative increase in total instrument error due to this cause is not great. Moreover, mechanisms with high transmission ratios are always more complicated, harder to make and less reliable. Therefore, if the sensitivity of an instrument is to be increased to reduce the indication error, its errors from other sources, including those of method, must be first determined. This analysis may show that no purpose would be served by increasing the sensitivity.

In particular cases, the same holds for an increase in the sensitivity of the instrument caused not by an increase in the transmission ratio of the transmitting mechanism, but by an increase in the displacement of the pickup, even if at the expense of a reduction in elasticity.

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CHAPTER II

DYNAMIC ERRORS IN INSTRUMENTS (OTHER THAN THOSE DUE TO VIBRATION)

Section 1. The Essence of Dynamic Errors

The previous Chapter was devoted to an examination of static errors in instruments. However, instruments may also present another type of error, known as "dynamic" error, the essence of which is the following:

As previously indicated, most instruments are built on the principle of measuring a force that is a function of the quantity to be measured. When the force to be measured by the instrument changes and for some time after becomes constant, many instruments provide readings that are not entirely accurate even if no static errors are present. In other words, instruments reveal a certain degree of inertia when the force to be measured changes. This takes place for the following reasons:

Primarily, in addition to the fact that the pickup of the instrument possesses a certain positional force of recovery (elastic or otherwise), which is what yields the measurement, the mechanism also possesses a certain mass on which some force must be expended for displacement, in order for accelerations to occur.

In the second place, damping (shock absorption) which is ordinarily necessary for a partial absorption of the errors under examination, often serves as one of the error sources.

Note: These phenomena of inertia and phenomena of retardation, occurring on changes in the instrument readings (similar to damper action) may be the result not only of mechanical forces of inertia and damping, but also

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temperature, electromagnetic resistances (due to capacitance in the system, self-inductance, magnetic polarity reversal inertia, etc.) and the like. In the present book, only the action of mechanical forces is treated. We note that, in the other types of inertia or retardation, the resultant instrument readings undergo distortion in a manner similar to that discussed in the present Section.

In addition, the dynamic error may include errors resulting from motion of the instrument mechanism relative to its housing, due to accelerations of variable magnitude or variable direction, occurring at the point of attachment of the instrument (as in vibrations).

If the instrument mechanism is considered a system with one degree of freedom, the equation for the forces affecting the behavior of the instrument (and reduced to some particular point in the mechanism) can be presented in the following fashion:

$$F_i + F_d + F_p \pm F_{fr} = f_1(t) + f_2(t), \quad (2.1)$$

where F_i is the total force of inertia due to the accelerated relative motion of any parts of the mechanism, reduced to a given point in the mechanism;

F_d is the total reduced force of damping in the system;

F_p is the total positional force of the mechanism (usually the force of the measuring spring);

F_{fr} is the reduced Coulomb friction in the mechanism;

$f_1(t)$ is the measured force, which is a function of time t ;

$f_2(t)$ are other additional forces, not subject to measurement and due to other factors. They are in some cases capable of affecting particular components of the instrument and, in this respect, of distorting its indications to some degree (i.e., the effect of accelerations in the

instrument mounting when some of its parts are out of balance, the effect of vibrations, etc.).

Note: The majority of instruments and analogous devices can be regarded as systems with one degree of freedom (with minor exceptions such as special accelerators, vibrometers, and similar devices), since an effort is usually made to provide unbroken kinematic connections between the various parts of the mechanism. Most often this is done by means of a hairspring. For this purpose, the latter need only be under adequate tension, i.e., the moment it creates must always be larger than the possible moment of the forces of inertia applied to the axis of the hairspring by parts whose meshing it facilitates.

If it were not for the forces of inertia F_i and of damping F_d , and if the supplementary distorting force of perturbation $f_a(t)$ were not present, the instrument readings would be determined from the equation

$$F_p \pm F_{fr} = f_1(t), \quad (2.2)$$

where F_p is a specific function of the displacement x of the pickup of an instrument and is functionally related to the readings of the instrument.

In this case, the instrument could only suffer errors of static measurement (one of which is the frictional error expressed by the force $\pm F_{tr}$ in eq.(2.2).

However, the presence of the forces F_i , F_d , and $f_2(t)$ produces the so-called dynamic error of the instrument.

Dynamic instrument errors, as we know, can be subdivided into two categories:

1. Errors due to the presence of the forces of inertia F_i and damping F_d , on any change in the measured quantity [when the quantity $f_1(t)$ changes], i.e., the error representing the difference between the instrument readings determined by the equation

$$F_i + F_d + F_p \pm F_{fr} = f_1(t), \quad (2.3) \text{ STAT}$$

and the indications determined by the equation

$$F_p \pm F_{fr} = f_1(t).$$

Errors of this type will be referred to below as Type I dynamic errors.

2. Errors due to the effect of supplementary perturbing forces $f_2(t)$, i.e., determined by the difference $F_p - F_{pc}$ in the equation

$$F_i + F_d + (F_p - F_{pc}) \pm F_{fr} = f_2(t), \quad (2.4)$$

where F_{pc} is a constant for the given value of the factor being measured, balanced by the force being measured $f_1(t)$, which, for the case of this equation, is considered to be equal to some constant magnitude C . Thus, eq.(2.4) shows the degree to which the instrument readings will differ from the indications governed by the equation $F_{pc} = C$, due to the effect of the force $f_2(t)$.

Hereafter, these errors will be referred to as Type II dynamic equations.

The second type of dynamic error differs in principle from the first in that here the error is not only a change in the instrument indications due to the presence of forces of inertia and damping in the equation of motion, but that the Type 2 dynamic error constitutes the totality of all deflections of the mechanism due to the force $f_a(t)$.

Note: It is obvious that when all forces acting on a mechanism are reduced to an axis O capable of rotation rather than to a point capable of moving linearly, eqs.(2.1), (2.3) and (2.4) can be replaced by equations for the reduced moments relative to the axis O , and the reasoning we applied to the equations of force can be applied to the equations of moments.

Section 2. Expressions for the Forces F_i , F_d and F_p in the Equations of Motion of an Instrument Mechanism

Expression for the Force of Inertia F_i . If the transmission ratio of the

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velocities of displacement of all parts of a mechanism of given mass or moment of inertia to the rate of displacement of the point to which the mass is reduced is constant, the expression for the force of inertia F_i will have the form

$$F_i = m_r \ddot{x},$$

where m_r is the so-called reduced mass of the mechanism;

x is the displacement of the point to which all forces acting on a mechanism are reduced.

However, if the forces are reduced to an axis, the moment of inertia will have the form

$$M_i = J_r \ddot{\varphi},$$

where J_r is the reduced moment of inertia,

φ is the angle of rotation of the axis to which the forces are reduced.

In this connection, it is known that

$$\left. \begin{aligned} m_r &= \sum \left[m_i \left(\frac{dy_i}{dx} \right)^2 \right] + \sum \left[J_n \left(\frac{d\alpha_n}{dx} \right)^2 \right]; \\ J_r &= \sum \left[m_i \left(\frac{dy_i}{d\varphi} \right)^2 \right] + \sum \left[J_n \left(\frac{d\alpha_n}{d\varphi} \right)^2 \right], \end{aligned} \right\} \quad (2.5)$$

where m_i is the mass of some part i , moving linearly;

dy_i is the elementary displacement of the center of gravity of that part, corresponding to the elementary displacement dx of the point to which the forces are reduced, or to the elementary angle $d\varphi$ of rotation of the axis to which the forces have been reduced.

J_n is the moment of inertia of some n^{th} part, rotating about a certain axis;

$d\alpha_n$ is the elementary rotation of this part, corresponding to the elementary displacement dx of the point to which the forces are reduced, or to ^{STAT} ~~the~~

elementary angle $d\varphi$ of rotation of the axis to which the forces have been reduced.

Should the transmission ratio of all parts of the mechanism be constant, we have a special case. In the general case, the expression for the total reduced force of inertia has the form

$$F_i = \left\{ \sum \left[m_i \left(\frac{dy_i}{dx} \right)^2 \right] + \sum \left[J_n \left(\frac{dz_n}{dx} \right)^2 \right] \right\} \ddot{x} + \left\{ \sum \left[m_i \left(\frac{dy_i}{dx} \right) \left(\frac{d^2 y_i}{dx^2} \right) \right] + \sum \left[J_n \left(\frac{dz_n}{dx} \right) \left(\frac{d^2 z_n}{dx^2} \right) \right] \right\} \dot{x}^2, \quad (2.6)$$

and the expression for the total reduced moment of inertia has the appearance

$$M_i = \left\{ \sum \left[m_i \left(\frac{dy_i}{d\varphi} \right)^2 \right] + \sum \left[J_n \left(\frac{da_n}{d\varphi} \right)^2 \right] \right\} \ddot{\varphi} + \left\{ \sum \left[m_i \left(\frac{dy_i}{d\varphi} \right) \left(\frac{d^2 y_i}{d\varphi^2} \right) \right] + \sum \left[J_n \left(\frac{da_n}{d\varphi} \right) \left(\frac{d^2 a_n}{d\varphi^2} \right) \right] \right\} \dot{\varphi}^2. \quad (2.6a)$$

Actually, the velocity of any part is

$$\dot{y}_i = \dot{x} \left(\frac{dy_i}{dx} \right)$$

or

$$\dot{a}_n = \dot{x} \left(\frac{da_n}{dx} \right).$$

The acceleration acquired by the part is

$$\ddot{y}_i = \ddot{x} \left(\frac{dy_i}{dx} \right) + \dot{x} \left(\frac{d^2 y_i}{dx^2} \right) \dot{x}$$

or

$$\ddot{a}_n = \ddot{x} \left(\frac{da_n}{dx} \right) + \dot{x} \left(\frac{d^2 a_n}{dx^2} \right) \dot{x}.$$

Therefore, to attain this acceleration, a force T must be applied to the STAT the value of which is

$$T = m_i \left[\ddot{x} \left(\frac{dy_i}{dx} \right) + \dot{x}^2 \left(\frac{d^2 y_i}{dx^2} \right) \right]$$

or a moment M , equal to

$$M = J_n \left[\ddot{x} \left(\frac{dz_n}{dx} \right) + \dot{x}^2 \left(\frac{d^2 z_n}{dx^2} \right) \right]$$

Thus, the reduced force T_r will be

$$T_r = T \frac{dy_i}{dx} = m_i \left[\ddot{x} \left(\frac{dy_i}{dx} \right)^2 + \dot{x}^2 \left(\frac{d^2 y_i}{dx^2} \right) \left(\frac{dy_i}{dx} \right) \right]$$

or

$$T_r = M \frac{dz_n}{dx} = J_n \left[\ddot{x} \left(\frac{dz_n}{dx} \right)^2 + \dot{x}^2 \left(\frac{d^2 z_n}{dx^2} \right) \left(\frac{dz_n}{dx} \right) \right]$$

Analogously, the reduced moment M_r will be equal to

$$M_r = m_i \left[\ddot{\varphi} \left(\frac{dy_i}{d\varphi} \right)^2 + \dot{\varphi}^2 \left(\frac{d^2 y_i}{d\varphi^2} \right) \left(\frac{dy_i}{d\varphi} \right) \right]$$

or

$$M_r = J_n \left[\ddot{\varphi} \left(\frac{dz_n}{d\varphi} \right)^2 + \dot{\varphi}^2 \left(\frac{d^2 z_n}{d\varphi^2} \right) \left(\frac{dz_n}{d\varphi} \right) \right]$$

Thus, without discussing the fact that in the case under consideration, the expressions $\frac{dy_i}{dx}$, $\frac{d^2 y_i}{dx^2}$, $\frac{dz_n}{dx}$, and $\frac{d^2 z_n}{dx^2}$ are permuted magnitudes or, in other words, are also various functions of x , we can make the following statements: In addition to terms constituting the product of acceleration $\ddot{x}$, the expression (2.5) also yields supplementary terms, proportional to the square of the velocity, resulting from the fact that this displacement causes a variation in the total mass of the parts. In this respect, it should be noted that, in some specific operating range of the mechanism, in which the signs of the derivatives $\frac{d\sigma_n}{dx}$ and $\frac{d^2 \alpha_n}{dx^2}$ or of the STAT

derivatives $\frac{dy_i}{dx}$ and $\frac{d^2y_i}{dx^2}$ do not change, the terms containing the square of the velocity as a factor, always have the same sign, i.e., they always tend to produce a further deflection of the mechanism in one or the other direction, regardless of the sign of the velocity x itself, exclusively as a function of the signs of the derivatives $\frac{dy_i}{dx}$ and $\frac{d^2y_i}{dx^2}$. Thus, these items are apparently capable of producing phenomena analogous to the so-called "Maxwell effect", which will be discussed below.

Usually, for an approximate examination of the question as to the dynamics of motion of mechanisms, if this motion is investigated within a narrow and limited range, these additional terms containing the factor x^2 are often neglected, as is the fact that the magnitudes $\frac{dy_i}{dx}$, $\frac{d\alpha_n}{dx}$, $\frac{dy_i}{d\varphi}$ and $\frac{d\alpha_n}{d\varphi}$ are not constant.

Note: In this connection, the familiar rule must be remembered that, in reduction of the mass of any element of a mechanism, it is also necessary to take into consideration all the motions of the element inseparably connected with the major motion.

In addition, cases may occur in which the displacement of some element results in an accelerated displacement of certain other masses. An example of this is the equation for the motion of a ball or an air bubble in a glide indicator, which will be examined in more detail below.

Expressions for the Damping Force F_d . When mechanisms are displaced, damping forces may arise both as a result of the resistance of the medium as the parts move or as a result of special damping devices. When the latter are employed (for example, a pneumatic shock absorber), the result sometimes is a very complex expression for the force F_d , applied in the form of an equation [analogous to eq.(2.1)] that cannot be solved by precise methods. This being the case, when such systems are analyzed, often approximately and usually only in very rough approximation, a

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simpler expression for the force F_d is used. Certain devices, however, such as liquid absorbers and dampers employing Foucault currents, create a damping force almost exactly proportional to the first power of the motion of the part being damped.

The resistance of the medium on displacement of parts of a mechanism often results in damping approximately proportional to the first power of the rate of motion of the part. But in certain cases this damping force, due to the resistance of the medium, may be proportional to the square of the velocity (of course, changing its sign as the direction of motion is reversed) or else it may be proportional to some velocity value higher than unity and lower than two.

In dampings proportional to some power of the velocity of motion of the element being damped, i.e., when the force of damping of that element is equal to $B_y \dot{y}^n$, where B_y is some dimensional coefficient and y is the displacement of the element, we have

$$B_y \dot{y}^n = B_y \left(\frac{dy}{dx} \right)^n \dot{x}^n.$$

To reduce this force to a point having a velocity of displacement x , we multiply the force by the quantity $\frac{dy}{dx}$. From this, we obtain

$$F_d = B_y \left(\frac{dy}{dx} \right)^{n+1} \dot{x}^n. \quad (2.7)$$

In the partial case, when $n = 1$, we have

$$F_0 = B_y \left(\frac{dy}{dx} \right)^2 \dot{x} = B_x \dot{x}.$$

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The factor k_r for the reduction of the damping factor*, is

$$k_{np} = \left(\frac{dy}{dx} \right)^{n+1}.$$

Thus, at $n = 1$, we obtain the same type of formula for the factor for reduction of the damping coefficient as for reduction of the mass or the coefficients of reduction of positional forces in the case of a constant transmission ratio $\frac{dy}{dx}$. In this case, regardless of whether the transmission ratio $\frac{dy}{dx}$ is constant or variable, we always obtain the same formula of reduction for the damping force, without supplementary terms, as we did in reducing the forces of inertia. Nor do we obtain the coefficient of reduction in a changed form, although, as we show below, such a form change takes place in the reduction of the coefficients of recovery of positional forces.

Equation for the Position Force, F_p . As already indicated, in many instruments the quantities to be measured with the particular instrument are determined indirectly, by measuring some force or moment constituting some specific function of the magnitude to be measured.

In the pickup, the force being measured is in most cases balanced by some other force whose magnitude depends on the point of application of the latter. Therefore, the position in the instrument of the point of application of the balancing force is a function of the magnitude of the force being measured.

A force, whose magnitude is a function $f(y)$ of the position y of the point of application of the force, i.e., whose value changes in a specific fashion with any displacement of the point of application of the force in the direction of application (or in the opposite direction), is known as a position force.

* This is the factor by which the damping coefficient B_y must be multiplied ^{STAT} in order to obtain the force F_d in eqs. (2.1), (2.2), or (2.4), expressed as $E_x \dot{x}$.

Spring units of various types are most frequently used as the position forces in instruments. Other types of position forces are also applicable, such as a gravity moment when a balance of some kind is displaced, a moment of recovery when the magnetic needle deviates from the direction of the magnetic pole, etc.

The function $f(y)$ of the position force is not linear in some cases and is a rather complex function of y in other cases (for example, in elastic metal diaphragms). However, in the usual type of spring, the ratio of the spring deflection and the resultant position force can be considered as linear, i.e., it has the form Cy , where C is a constant (spring rigidity).

The ratio of an infinitesimal change dP in the position force P to the corresponding infinitesimally small displacement dy of the point of application of the force in a direction opposite to that of the attacking force, i.e., the derivative

$$\frac{d[f(y)]}{dy} = \frac{dP}{dy}$$

will be denoted as the recovery factor of the position force. Thus, in elastic elements the recovery factor is the elasticity or rigidity of the elastic element.

In the general case, the recovery factor of the position force may be inconstant, i.e., it may also be some function of the displacement y of the point of application of the force. As indicated above, in this case only a displacement, either in the direction in which the force is acting or in the opposite direction, is meant here.

In addition to the foregoing, let us adopt the following: In an expression representing the position force in the form of some function $f(y)$ of the displacement y of the point of application of the force [or in a suitable graph or Table, if the $f(y)$ function is not given in analytical form but in the form of a graph or by means of a Table], we will consider the displacement y to be positive in the direction opposite to the action of the force.

Analogously, a moment is called positional if its magnitude depends on the angle of rotation of the part to which the moment is applied, i.e., when the moment $M(\varphi)$ is some function of the angle of rotation φ of the part. The coefficient of recovery of the position moment is the term applied to the $\frac{dM}{d\varphi}$ ratio, where dM is the simple increase in moment, corresponding to the simple rotation $d\varphi$ of the given part in a direction opposite to that in which the moment is acting.

In certain cases, not only the equalizing force but also the force being measured are covered by the definition for the position forces, offered above, since they are specific functions of the quantity being measured. By way of an example, we cite the centrifugal force which can be measured, for example, in a centrifugal tachometer and which is not only a function of the rate of rotation but also a function of the radius of rotation, i.e., of the distance from the point of application to the angle of rotation. However, if the ratio of the magnitude of the force being measured to the position of the point of application is not identical with the ratio of the magnitude of the force being balanced to the position of the point of application, we may still obtain, for each dimension in the parameter being measured, its specific point of equilibrium in the mechanism.

In some mechanisms, several position forces act on one and the same point in the mechanism. In other mechanisms, several position forces act on different parts. In this case, it is also often necessary to determine the total coefficient of recovery of all the position forces, relative to some particular point or axis of the mechanism. This requires either simply a summing up of the recovery factors, if the position forces have been applied to a single point in the mechanism, or a reduction of all these recovery factors before performing the addition.

If the transmission ratio of the point of application of the position force to the point to which the forces are reduced is not constant, the reduced force $F_p = F(z)$, constituting a function of the displacement z of the point of

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reduction, may be a function of a type entirely different from that of the $f(y)$ function of the unreduced force (in this case, y is the displacement of the point of application of force). We will give a more detailed discussion of this question in a separate Chapter on the reduction of position forces, since this point is of considerable significance in the static analysis of mechanisms.

Thus, the nonlinearity of the function $F_p = F(z)$ may result both from the nonlinearity of the position force itself (as, for instance, in elastic metal diaphragms) and from the inconstancy of the transmission ratio $\frac{dy}{dz}$ in cases in which it is necessary to apply a position force.

However, as already indicated, the $F(z)$ function is virtually linear in many instruments. The following may be noted in this respect: Dynamic errors are often examined under conditions of minor changes in the quantities being measured by the instrument. This is due to the fact that, in many instruments including a majority of aircraft instruments, the quantities measured are usually unable to undergo major changes at such a rate as to make this a source of noticeable dynamic errors. However, although this phenomenon may occur in individual instruments, it is rare and of very short duration, during which time no readings of the particular instruments should be taken. Individual special instruments, whose readings are used during the occurrence of such phenomena (i.e., transfer instruments, etc.) usually have almost exactly the linear characteristic of the pickup element.

Therefore, for purposes of a sample calculation of its dynamic errors, it is often assumed, in first approximation, that

$$F_p \approx (F_p)_{z=z_0} + \left(\frac{dF_p}{dz} \right)_{z=z_0} \Delta z,$$

where z_0 is some mean value z in the range under examination (i.e., here, in an expansion of the quantity F_p into a Taylor series, the discussion will be limited to a consideration only of the first two terms in the series).

Accordingly, in this operation, using the function $f_1(t)$ in eq.(2.3), it frequently happens that not the entire force measured by the instrument but only the deviation of the magnitude of this force from some mean value $f_{1\text{mean}}(t)$ in the range of the instrument readings under examination, corresponding to the value z_0 of the displacement of the point of application of the reduced position force of the instrument [in accordance with the equation $(F_p)_{z=z_0} = f_{1\text{mean}}(t)$] is obtained. Therefore, when so designated, the total magnitude of the force being measured is $f_{1\text{mean}}(t) + f_1(t)$. The magnitude $f_{1\text{mean}}(t)$ is then contracted in eq.(2.3) with the term $(F_p)_{z=z_0}$ on the left side of the equation; the resultant equation will differ from eq.(2.3) only in that the term F_p is replaced by the term $\left(\frac{dF_p}{dz}\right)_{z=z_0} \Delta z$. The quantity Δz is designated only by z or by some new unknown x , so that the term F_p in eqs.(2.1), (2.2), and (2.3) takes on the form of Cx . Ordinarily, the quantity C is taken approximately as a constant equal to $\left(\frac{dF_p}{dz}\right)_{z=z_0}$, which we denote as the position force recovery factor. This is more likely to occur since, in many instruments, this quantity changes very slowly in general with any change in z . When elastic elements are used, the source of the position force C is the rigidity of the elastic element.

As far as the inertia and damping terms, F_i and F_d in these equations are concerned, the terms $\ddot{z}$ and $\dot{z}$ can be simply replaced by the corresponding terms $\Delta \ddot{z}$ and $\Delta \dot{z}$, or by new designations for these latter, since z and Δz differ by a constant. We note, further, that virtually all instruments and similar systems present a positive total recovery factor C , since each reading of the instrument corresponds to some specific position of stable equilibrium of its mechanism. Therefore, in compiling eq.(2.1), the term $F_p = Cx$ is written with the same sign as the terms $\ddot{z}$ and $\dot{z}$ (considering the values m , B , and C to be positive). STAT

Conventional Form for Writing Equations for the Forces Acting on an Instrument

Mechanism. Thus, the forces F_1 and F_d in the equilibrium of forces, acting on an instrument mechanism, can usually be replaced by simpler expressions of the type of $m_r \ddot{x}$ and Cx , where m_r is the reduced mass of the system.

In addition, the force of damping of the system, for the sake of simplicity, assumed to be directly proportional to the first power of the velocity $\dot{x}$, i.e., the force F_d is expressed in the form of $B\dot{x}$, where B is the damping factor of the system. In individual cases this is not quite accurate, but for many instruments this simplification has comparatively little effect on the result of the investigation, since the degree of damping is, in general, very small.

These simplifications in the equations for the forces acting on instrument mechanisms are further justified by the fact that, in many instruments with elastic elements for measuring forces of one kind or another, the efforts to overcome dynamic errors are reduced, as demonstrated below, to a simple rule. All that is required is that the natural period of vibration of the instrument mechanism differ, as far as possible from the period of the perturbing forces $f_2(t)$, if these forces are periodic. Therefore, a minor inaccuracy is of no great significance in studies of the oscillatory motion of an instrument mechanism.

Thus, in studying the dynamic errors of instruments considered as systems with one degree of freedom, the equation for the forces acting on the instrument usually is reduced in the general case, to the linear form, namely,

$$\begin{aligned} \text{or} \quad & \left. \begin{aligned} m_r \ddot{x} + B\dot{x} + Cx + F_{gr} &= f_1(t) + f_2(t) \\ J_r \ddot{\varphi} + H\dot{\varphi} + S\varphi + M_{gr} &= f_1(t) + f_2(t), \end{aligned} \right\} \quad (2.8) \end{aligned}$$

if all the forces and moments acting on the mechanism are reduced to an axis in the mechanism rather than to some point in linear motion. In this last equation, H is the reduced specific damping moment of the mechanism (the damping factor), S the specific position moment (position moment recovery factor), and M_{gr} the reduced

frictional moment.

The force of Coulomb friction $^+F_{fr}$, which is usually regarded as constant in absolute magnitude but varying in sign on reversals in the direction of motion of the mechanism, will not complicate the solution of the above equation only when the forms taken by the forces $f_1(t)$ and $f_2(t)$ are such that the motion of the mechanism can be regarded as always being in the same direction, with the result that the friction force does not change its sign. However, as already indicated, this case is only of interest in only a few instances, in studies of dynamic errors in instruments.

However, if in the process under examination the direction of motion is variable, a consideration of the $^+F_{fr}$ force in the above equation markedly complicates any investigation of the forced motion of the mechanism. Therefore, in approximate analysis, the force of the Coulomb friction is usually neglected for purposes of simplification. In most instruments, this type of simplification has no significant effect on the results of the investigation, since the friction in the instruments is quite low. In addition, dynamic errors can often be prevented by making an allowance, in instrument design and calculations, for the natural oscillation frequency in the system.

Furthermore, the effect of Coulomb friction can be approximately compensated by a suitable increase in the factor B of the term $B\dot{x}$ in the equation in question.

Only in special cases (as in vibrographs and accelographs) is a more detailed analysis of dynamic errors required. In this case, a more precise determination of the effect of Coulomb friction may be necessary. Unfortunately, this is not always easy to attain. For the special case in which the form of the function for the perturbing force $f_2(t)$ is harmonic, and under conditions in which the natural frequency of oscillation of the mechanism is less than the frequency of the disturbing force, methods are in existence for determining the law governing steady-state forced vibrations in the mechanism, with allowance for the effect of Coulomb friction.

However, since these methods are applicable only to a relatively small number of special instruments, we will discuss this question in a special paragraph at the end of the Chapter on dynamic instrument error.

Section 3. Natural Vibrations in a Mechanism in the Absence of Forced Motion

Let us examine a few special cases covered by eq.(2.8). Let the force to be measured $f(t)$ have become constant, i.e., let $f(t) = R$ where R is a constant, while the mechanism is assumed to have left the position corresponding to this force (or has not yet become fixed in that position). In that case, we have the equation

$$m\ddot{x} + B\dot{x} + Sx + F_r = R. \quad (2.9)$$

This type of equation is readily solved. For example, if $F_{fr} = 0$, the solution of the equation in a case in which $\frac{S}{m} > \left(\frac{B}{2m}\right)^2$ will be

$$x = \frac{R}{S} + e^{-\frac{B}{2m}t} \left(c_1 \sin \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t + c_2 \cos \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t \right). \quad (2.10)$$

The constants of integration c_1 and c_2 are determined from the initial conditions. Thus, for a case in which, at $t = 0$, $x = x_0$ and $\dot{x} = 0$, we have

$$x_0 - \frac{R}{S} = c_2$$

Since

$$= e^{-\frac{B}{2m}t} \left\{ -\frac{B}{2m} \left[c_1 \sin \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t + \right. \right.$$

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$$+ c_2 \cos \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t + \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} \left[c_1 \cos \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t - c_2 \sin \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t \right],$$

it follows that

$$0 = -\frac{B}{2m} c_2 + \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} c_1.$$

Consequently,

$$x = \frac{R}{S} + e^{-\frac{B}{2m}t} \left(x_0 - \frac{R}{S} \right) \left[\frac{\frac{B}{2m}}{\sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2}} \sin \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t + \cos \sqrt{\frac{S}{m} - \left(\frac{B}{2m}\right)^2} t \right]$$

and, if, as is generally taken to be the case, the magnitude $\frac{S}{m}$ is denoted by

k^2 and $\frac{B}{2m}$ by h , we obtain

$$x = \frac{R}{S} + e^{-ht} \left(x_0 - \frac{R}{S} \right) \left[\frac{h}{\sqrt{k^2 - h^2}} \sin \sqrt{k^2 - h^2} t + \cos \sqrt{k^2 - h^2} t \right]. \quad (2.11)$$

Thus, we are dealing with steadily damped vibrations about a position of STAT

equilibrium defined by the equation $x = \frac{R}{S}$. The vibration period T , is equal to $\frac{2\pi}{\sqrt{k^2 - h^2}}$. The value of $\sqrt{k^2 - h^2}$ is the angular velocity of these vibrations, in radians per second. The magnitude of each successive extreme position of the point in the instrument mechanism under examination is smaller, by a factor of $e^{-\frac{T}{h_2}}$ than the former extreme deviation in the opposite direction. The $\frac{h}{k}$ ratio describes the degree of damping.

Damping of the vibrations takes place in the order indicated in the graph in Fig.7.

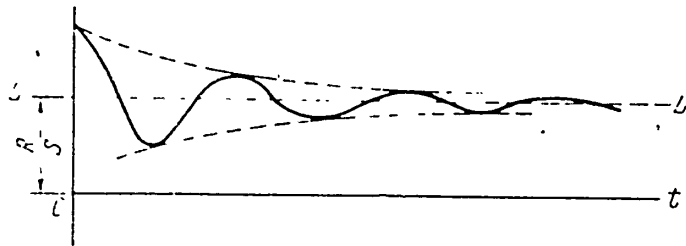


Fig.7 - Character of Decay of Free Vibrations under Damping,
Proportional to the First Power of the Velocity

However, if friction proportional to the first power of the velocity (term $R\dot{x}$) is lacking while Coulomb friction F_{fr} is present, and if, for the sake of simplicity, this absolute is considered a constant, the motion of the point under examination can be described by the graph in Fig.8.

In this graph, the added broken lines EE_1 and NN_1 have been drawn parallel to the axis t , at a distance $\Delta x_{fr} = \frac{F_{fr}}{S}$ from the straight line LL_1 . Here the space from $\frac{R}{S} - \Delta x_{fr}$ to $\frac{R}{S} + \Delta x_{fr}$ is the zone of sluggishness of the instrument mechanism.

The curves are plotted as follows: For the first half-cycle from point A_0 to point A_1 , the half-cycle of an ordinary sinusoid is used, in which the center line (the geometric locus of the points of symmetry of the half-cycles) is the line EE_1 . Thus, the amplitude of this sinusoidal half-cycle is

$$x_0 - \Delta x_{fr} = x_0 - \frac{F_{fr}}{S}.$$

Further,

$$A_0B_0 = A_1B_1 \text{ and } A_1D_1 = x_0 - 2\Delta x_{fr},$$

For the second half-cycle, from point A_1 to A_2 , use is made of the half-cycle of sinusoids, whose center line is the line NN_1 . Thus,

$$A_1C_1 = A_2C_2 \text{ and } A_2D_2 = x_0 - 4\Delta x_{fr}$$

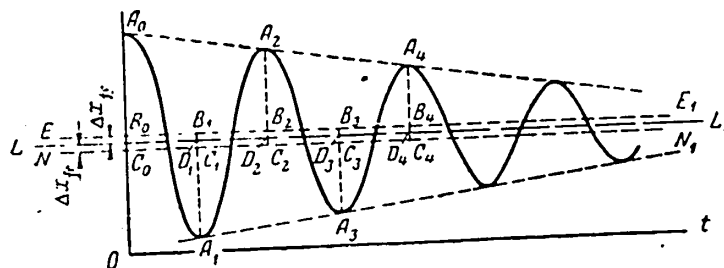


Fig.8 - Character of Attenuation of Free Vibrations due to Presence of Dry Friction

In the following half-cycle, from A_2 to A_3 , the center line is also the line EE_1 , while in the next half-cycle, from A_3 to A_4 , it is the line NN_1 , etc. Thus, $A_0B_0 = A_1B_1$; $A_2B_2 = A_3B_3$; $A_4B_4 = A_5B_5$, etc., and $A_1C_1 = A_2C_2$; $A_3C_3 = A_4C_4$, etc. STAT

In the given case, the frequency is $k = \sqrt{\frac{S}{m}}$, while the period of vibration is

$$T = 2\pi \sqrt{\frac{m}{S}}$$

When, in a case of this kind of motion, one of the extreme positions x_n differs so little from $\frac{R}{S}$ that $|x_n - \frac{R}{S}| < \frac{F_{fr}}{S}$, motion will cease entirely.

We may use analogous reasoning in cases in which both Coulomb friction and damping occur. As a result we have the motion shown by the curve in Fig.9. In this curve, the segment from A_0 to A_1 is plotted on the basis of the equation

$$x = \frac{R}{S} + \frac{F_{tp}}{S} + e^{-ht} \left(x_0 - \frac{R}{S} - \frac{F_{tp}}{S} \right) \times \left[\frac{h}{\sqrt{k^2 - h^2}} \sin \sqrt{k^2 - h^2} t + \cos \sqrt{k^2 - h^2} t \right]. \quad (2.12)$$

The point A_1 corresponds to the position of the vibrating point after the passage of a period of time equal to a half-cycle of vibration (counting from the beginning of motion).

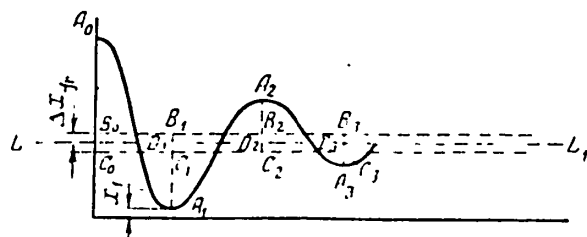


Fig.9 - Character of Decay of Free Vibrations in the Presence of Dry Friction and Damping, Proportional to the Velocity

At this instant the position of the point is determined from the equation

$$A_1 B_1 = e^{-h \frac{T}{2}} A_0 B_0,$$

where T is the half-cycle of vibration of the point, i.e.,

$$T = \frac{2\pi}{\sqrt{k^2 - h^2}}$$

and

$$A_0 B_0 = x_0 - \frac{R}{S} - \frac{F_{fr}}{S}.$$

Further (see Fig.9)

$$\left| x_1 - \frac{R}{S} \right| = A_1 D_1 = A_1 B_1 - \Delta x_{fr} = A_1 B_1 - \frac{F_{fr}}{S}.$$

Analogously, the curve segment from A_1 to A_2 is plotted in accordance with the equation

$$x = \frac{R}{S} - \frac{F_{fr}}{S} + e^{-h \left(t - \frac{T}{2} \right)} \left[x_1 - \frac{R}{S} + \frac{F_{fr}}{S} \right] \times \quad (2.13)$$

$$\times \left[\frac{h}{\sqrt{k^2 - h^2}} \sin \sqrt{k^2 - h^2} t + \cos \sqrt{k^2 - h^2} t \right]$$

while the point A_2 is determined from the equation

$$A_2 C_1 = e^{-h \frac{T}{2}} A_1 C_1.$$

Proceeding on similar reasoning, we can plot the entire curve for the segments A_0 to A_1 , A_1 to A_2 , A_2 to A_3 , etc., and obtain the total curve for the motion of the point under examination.

If, at the outset, when $t = 0$, the velocity $\dot{x}$ equals not zero but some specific

magnitude, the solution will be analogous in all cases under examination (depicted in the curves of Figs. 7, 8, and 9). The only change is a slight variation in the integration constants, at an incipient change in the initial conditions. Naturally, in this case the first half-cycles of the graphs in Figs. 8 and 9 are incomplete. In general, however, the motion will be similar.

All these cases of vibration in the mechanism of an instrument prior to damping, are natural oscillations of the instrument mechanism.

For all the cases discussed, the magnitude $\frac{2\pi}{\sqrt{\frac{S}{m}}}$ is termed the period of free

undamped vibration of the mechanism.

In cases where $\frac{S}{m} \leq \left(\frac{B}{2m}\right)^2$, an aperiodic motion to a position of equilibrium occurs.

This case will not be discussed here.

Section 4. Forced and Free Motion of the Mechanism for a Variable Function $f(t)$

As indicated above, in examining this question, the term in eq.(2.8) representing the effect of Coulomb friction, will be disregarded, for reasons of simplicity. Then, in the general case - for various forms of the function $f(t)$ - the equation

$$m\ddot{x} + B\dot{x} + Sx = f(t) \quad (2.14)$$

can be solved by the method of variation of integration constants. As we know, this method consists in the fact that, at the outset, we take a solution of a homogeneous equation [i.e., the value $f(t)$ in the equation is replaced by zero] so that the integration constants in the homogeneous equation are no longer considered constants but various functions of t , and are selected so as to satisfy eq.(2.14). We

know that for an equation of the second order [and for a case in which $\frac{S}{m} > \left(\frac{B}{2m}\right)^2$], this method makes it possible to obtain a solution of the equation, in the following form:

$$x = \frac{e^{-ht}}{\sqrt{k^2 - h^2}} \left\{ \sin \sqrt{k^2 - h^2} t \int e^{ht} \frac{f(t)}{m} \cos \sqrt{k^2 - h^2} t dt - \cos \sqrt{k^2 - h^2} t \int e^{ht} \frac{f(t)}{m} \sin \sqrt{k^2 - h^2} t dt \right\}, \quad (2.15)$$

where $k^2 = \frac{S}{m}$ and $h = \frac{B}{2m}$.

Here the integrals are indefinite so that, in integrating this equation, two integration constants are obtained. In other words, in addition to an expression depending on the form of the function $f(t)$, a term of the following form is obtained:

$$e^{-ht} (c_1 \sin \sqrt{k^2 - h^2} t + c_2 \cos \sqrt{k^2 - h^2} t),$$

in which c_1 and c_2 are integration constants. The general appearance of this last expression does not depend on the form of function $f(t)$, but of course the quantitative magnitudes of the integration constants c_1 and c_2 depend both on the initial conditions and on the form of the function $f(t)$. It is easy to see that this last expression consists of the same free oscillations discussed in the preceding section.

Thus, the general motion of a point in a mechanism for which an equation of motion has been written, consists of the so-called forced motion [whose expression is determined by the form of the function $f(t)$] and of the free oscillation of the system, both superimposed on the forced vibrations. STAT

Expression (2.15) is often also written in the following form:

$$x = \frac{e^{-ht}}{\sqrt{k^2 - h^2}} \left\{ \sin \sqrt{k^2 - h^2} t \int_a^t e^{h\tau} \frac{f(\tau)}{m} \cos \sqrt{k^2 - h^2} \tau d\tau - \right. \\ \left. - \cos \sqrt{k^2 - h^2} t \int_b^t e^{h\tau} \frac{f(\tau)}{m} \sin \sqrt{k^2 - h^2} \tau d\tau \right\}.$$

Here the integration limits a and b are the integration constants. It is quite obvious from the form of the equation that its integration will yield an expression of exactly the same type as an integration of eq.(2.15), i.e., the same equation for forced motion on which the natural oscillations of the system, referred to above, are superimposed. The forced motion of the mechanism depends on the appearance of the function $f(t)$ and on the parameters of the system (on the quantities m , D , and S). In natural oscillations, the integration constants depend on the initial conditions, on the expression for $f(t)$, and on the parameters of the system. For the rest, the form of the natural oscillations is dependent solely on the parameters of the system.

In a number of special cases of $f(t)$ shape, simpler methods for the solution of eq.(2.14) and simpler final equations for forced vibrations exist. These solutions have been described in several publications so that they will not be discussed here. As an example, let us cite only a case in which the $f(t)$ function is harmonic, i.e., has the form

$$f(t) = A \sin(\omega t + \epsilon),$$

where ω is the angular velocity of this function in radians per second (its frequency expressed in periods per second, equal to $\frac{\omega}{2\pi}$);

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A is the oscillation amplitude of the perturbing force, or the force to be

measured;

ε is some constant dimensionless (angular) quantity.

Then, as we know, the solution of the equation will be

$$x = \frac{A}{m \sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} \sin(\omega t + \varepsilon - \delta) + e^{-ht} (c_1 \sin \sqrt{k^2 - h^2} t + c_2 \cos \sqrt{k^2 - h^2} t), \quad (2.16)$$

where δ is a constant equal to

$$\delta = \arctg \frac{2\omega h}{k^2 - \omega^2}. \quad (2.17)$$

In writing the second term of the right side of eq.(2.16), we will only discuss the case in which $\frac{S}{m} > \left(\frac{B}{2m}\right)^2$.

Note: In this instance, should the magnitude $k^2 - \omega^2$ be negative (when $\omega > k$), the angle δ will be within the limits of $\frac{\pi}{2}$ and π .

The first term in eq.(2.16) represents forced motion of the point for which eq.(2.14) had been written, consisting, in the given case, of so-called "forced oscillations". Thus, forced oscillations are of the same frequency as the perturbing force $A \sin(\omega t + \varepsilon)$, but their amplitude in the general case differs from that of so-called "static" deflections of points corresponding to the acting force $f(t)$. In addition, forced oscillations are shifted in phase relative to the oscillations of the perturbing force (by an angle δ).

We call static that deflection which would occur if we could ignore the velocity terms $\ddot{mx}$ and $B\dot{x}$ in eq.(2.14). The ratio of the amplitude of forced oscillations to the amplitude of static deflections is termed the dynamic susceptibility factor of the system and denoted by the symbol λ . The amplitude of the static deflections is obviously equal to $\frac{A}{S}$. Consequently, STAT

$$\lambda = \frac{S}{m \sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} = \frac{k^2}{\sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{k^2}\right)^2 + 4 \frac{\omega^2}{k^2} \frac{h^2}{k^2}}} \quad (2.18)$$

We see from eq.(2.18) that the magnitude of the dynamic susceptibility is a function of the ratio of the frequencies ω and k , and of the degree of damping $\frac{h}{k}$.

In some cases, the dynamic increase may reach excessive magnitudes. When the degree of damping is not very great, the maximum dynamic susceptibility is that occurring when the ratio is close to unity. This case is known as resonance.

Figure 10 presents curves showing the magnitude of the dynamic susceptibility factor λ relative to the magnitude of the $\frac{\omega}{k}$ ratio for various degrees of damping.

We see that when the damping degree $\frac{h}{k}$ reaches a magnitude at which $h^2 = 0.5k^2$ ($h \approx 0.70k$), the curves no longer show a maximum at values of $\frac{\omega}{k} > 0$ and the dynamic susceptibility factor does not exceed unity at any values of the $\frac{\omega}{k}$ ratio. In other words, there is no increase in amplitude, even in the case of resonance.

Actually, to find the maximum value for the expression, we determine the minimum for the value of the radicand N

$$N = \left(1 - \frac{\omega^2}{k^2}\right)^2 + 4 \frac{\omega^2}{k^2} \frac{h^2}{k^2}.$$

Thus we obtain

$$\frac{dN}{d\left(\frac{\omega}{k}\right)} = -4 \left(1 - \frac{\omega^2}{k^2}\right) \frac{\omega}{k} + 8 \frac{\omega}{k} \frac{h^2}{k^2}.$$

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If we equate this derivative to zero, we will have

$$\frac{\omega}{k} \left(\frac{\omega^2}{k^2} - 1 + 2 \frac{h^2}{k^2} \right) = 0.$$

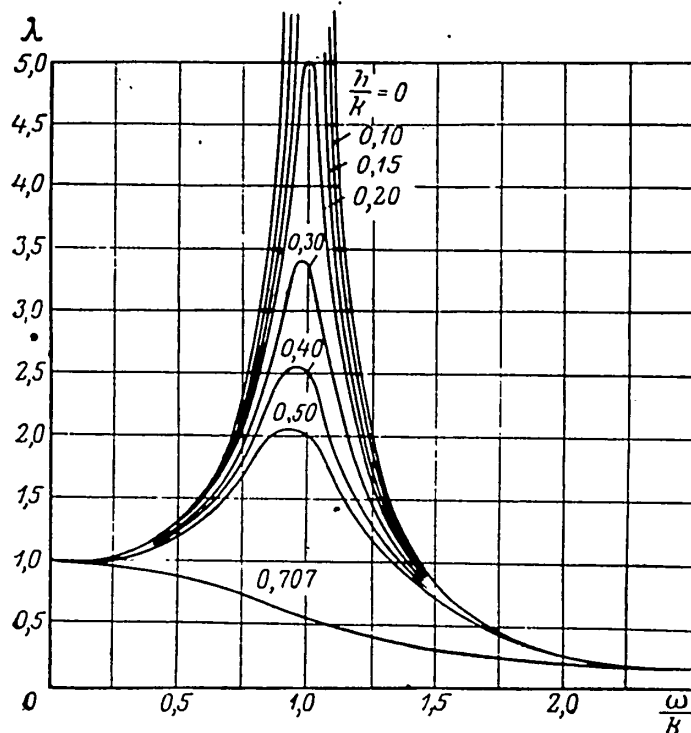


Fig.10 - Ratio of the Dynamic Susceptibility Factor, λ to the $\frac{\omega}{k}$ Ratio between Forced and Natural "Undamped" Oscillations as a Function of the Damping $\frac{h}{k}$.

It is obvious that the expression in parentheses may be equated to zero only to the point at which $2 \frac{h^2}{k^2} < 1$ in which case we will have a minimum, as

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$$\frac{d^2 N}{d\left(\frac{\omega}{k}\right)^2} = -4\left(1 - \frac{\omega^2}{k^2}\right) + 8\frac{\omega^2}{k^2} + 8\frac{h^2}{k^2};$$

where

$$\frac{\omega^2}{k^2} - 1 + 2\frac{h^2}{k^2} = 0,$$

so that

$$\frac{d^2 N}{d\left(\frac{\omega}{k}\right)^2} = 8\frac{\omega^2}{k^2},$$

i.e., $\frac{d^2 N}{d\left(\frac{\omega}{k}\right)^2}$ is a positive quantity.

Thus, the dynamic susceptibility factor reaches a maximum at $\frac{\omega}{k}$ values greater than zero, only when $h^2 < \frac{k^2}{2}$. In addition, we see that, at low magnitudes of the $\frac{\omega}{k}$ ratio, the dynamic susceptibility ratio is close to unity.

Here we do not derive formulas for "forced motion" in the case of other forms of $f(t)$ functions, since such formulas may be found in other texts on the pertaining subjects.

In the expression for the natural vibrations of a system, the integration constants are, as we know, determined from the initial condition, i.e., from the initial position of x_0 , the point for which eq.(2.14) was written, and from the initial velocity $\dot{x}_0$, at a given moment in time, in the continuation of which the $f(t)$ expression is a specific function of t of the type examined above. STAT
Usually, the initial moment of action of the force $f(t)$ is used. In all cases, i.e., whatever the form of the $f(t)$ function, these integration constants are

determined after all the factors in the expression for forced motion have been found, since the magnitude and phase of the former also depend on that expression (for example, in the case of forced vibrations they depend on the parameters of this forced vibration).

Natural oscillations (or natural aperiodic motions, in the case of an aperiodic system) develop at the instants of change in the analytical expression of the function $f(t)$, including the instants when it disappears or converts to a constant. In general, natural oscillations (or natural aperiodic motion) develop as a result of the fact that, at the initial moment of action of the perturbing forces, and in accordance with the given function $f(t)$, the position and rate of motion of the point to which all the forces are reduced no longer coincide with the position and motion which this point should have in accordance with the expression for forced motion.

However, if, at the initial time, the position and rate of motion of the point accidentally coincides with the position and motion it should have in accordance with the law of forced motion, no natural oscillations will arise - or natural aperiodic motion in the case of $\frac{S}{m} \leq \left(\frac{B}{m}\right)^2$ - so that eqs.(2.15) and (2.16) will retain only terms representing forced motion. Among other things, this derives directly from the method of determining the quantitative magnitude of the integration constants c_1 and c_2 in eqs.(2.15) and (2.16) or in the expressions for aperiodic motion, proper to aperiodic systems.

When damping is present (Bx terms), the natural frequencies gradually decay - if the $f(t)$ function of the perturbing force has not changed in form - and after a while only forced motion remains. This outcome is facilitated to some degree by Coulomb friction, in a manner analogous to the prior case in which only natural STAT vibration was present (see the curves in Figs.7 and 8).

It must be stated that, in the case of forced motion in which the direction of

motion changes, the existence of Coulomb friction may also give rise to natural oscillations, with such oscillations being excited anew at each change in the direction of motion. However, these are not the natural oscillations that depend on the initial conditions when the function $f(t)$ first comes into play, but special natural oscillations. Their magnitude and phase depends solely on the nature of forced motion and the parameters of the system. This latter phenomenon will be explained in a special Section on forced vibrations in the presence of Coulomb friction.

Section 5. Type I Dynamic Errors

Thus, we identify as Type I dynamic errors the difference between solutions of the equation

$$m\ddot{x} + B\dot{x} + Sx \pm F_{fr} = f_1(t)$$

and the equation

$$Sx \pm F_{fr} = f_1(t),$$

where $f_1(t)$ represents the force subject to measurement in the given instrument. As indicated above, we examine here only those systems to which application of linear equations of the given type is permissible, in first approximation. In this case, for purposes of simplification, we neglect the force of Coulomb friction $\pm F_{tr}$. When the question is examined in this fashion, Type I dynamic errors take the form of the difference between expressions (2.15) - or (2.16) - and the expression for the so-called "static" deflections of the point to which all the forces in the mechanism are reduced, equal to $\frac{f_1(t)}{S}$.

This difference is divided into two terms, namely: the difference between the positions determined by "forced" motion and "static" deflection of the point, and deflection due to natural vibrations in the system. If, for example, we take a case harmonic in form, using the function

$$f_1(t) = A \sin(\omega t + \epsilon),$$

then the deflection x_b of the point due to forced motion will be

$$x_b = \frac{A}{S} \lambda \sin(\omega t + \epsilon - \delta),$$

where λ is the dynamic susceptibility factor of the system;

δ is the phase difference between forced and disturbing oscillations.

The static deflection x_{st} from the point will be, in the given instance,

$$x_{st} = \frac{A}{S} \sin(\omega t + \epsilon).$$

From this we find that the difference between the two deflections is

$$\begin{aligned} x_b - x_{st} &= \frac{A}{S} [\lambda \sin(\omega t + \epsilon - \delta) - \sin(\omega t + \epsilon)] = \\ &= \frac{A}{S} [\lambda \sin(\omega t + \epsilon) \cos \delta - \lambda \cos(\omega t + \epsilon) \sin \delta - \sin(\omega t + \epsilon)] = \\ &= \frac{A}{S} [(\lambda \cos \delta - 1) \sin(\omega t + \epsilon) - \lambda \sin \delta \cos(\omega t + \epsilon)] = \quad (2.19) \\ &= \frac{A}{S} \sqrt{(\lambda \cos \delta - 1)^2 + \lambda^2 \sin^2 \delta} \sin(\omega t + \epsilon - \mu) = \\ &= \frac{A}{S} \sqrt{1 + \lambda^2 - 2\lambda \cos \delta} \sin(\omega t + \epsilon - \mu), \end{aligned}$$

where

$$\mu = \arctg \frac{\lambda \sin \delta}{\lambda \cos \delta - 1}.$$

Note: Here, in the case of a negative magnitude of the denominator

$\lambda \cos \delta - 1$, the angle μ is greater than $\frac{\pi}{2}$ and smaller than π .

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Thus, at the instant when $\omega t + \epsilon - \mu = (2n + 1) \frac{\pi}{2}$, where n is any integer, i.e.,

when

$$t = \frac{(2n+1) \frac{\pi}{2} - \epsilon + \mu}{\omega},$$

the difference between the quantities x_B and x_{st} will be its maximum and reach the value of

$$(x_B - x_{st})_{\max} = \frac{A}{S} \sqrt{(1 - \lambda \cos \delta)^2 + \lambda^2 \sin^2 \delta}.$$

Thus, when the function $f_1(t)$ is harmonic, the difference $x_B - x_{st}$ also varies harmonically with time, with the same frequency ω as the perturbing function; however, the vibrations of the difference $x_B - x_{st}$ are shifted in phase both relative to oscillations of the perturbing force (by an angle μ) and relative to forced vibrations (by an angle $\mu - \delta$).

Aside from the magnitude $x_B - x_{st}$, natural oscillations, so long as they have not yet decayed, also cause certain additional deflections, which are usually difficult to allow for. These errors are added algebraically to the error resulting from the presence of a difference between the deviations x_B and x_{st} .

In this situation, the mechanism with natural vibrations does not immediately come to rest in a position of equilibrium even when $f(t)$ is a constant. Therefore, a certain degree of damping is required to absorb the natural vibrations. It is clear that, when the degree of damping increases, the natural vibrations of the system damp more rapidly. We know that a system may have natural vibrations only to the point at which $\left(\frac{B}{2m}\right)^2 < \frac{S}{m}$. However, if $\left(\frac{B}{2m}\right)^2$ is equal to or greater than $\frac{S}{m}$, an aperiodic natural motion will occur. When this is the case, the system taken out of equilibrium will, if $f(t)$ is a constant, return asymptotically to a position of STAT equilibrium, but not proceed past it. However, systems with so-called hundred per-

cent damping [when $\left(\frac{B}{2m}\right)^2 = \frac{S}{m}$] or overdamped systems [when $\left(\frac{B}{2m}\right)^2 > \frac{S}{m}$] are used rarely, since aperiodic systems are usually quite inert, i.e., the time required for them to reach equilibrium is quite high. Non-aperiodic systems are therefore more frequently employed. In many instruments - compasses, for example - the type of damping most commonly employed is that in which

$$\left(\frac{B}{2m}\right)^2 = 0.5 \frac{S}{m}$$

(meaning that $h = \frac{B}{2m} = \sqrt{0.5} \sqrt{\frac{S}{m}} \approx 0.707 k$), since the results obtained are more desirable. This is generally known as 70% damping.

However, in addition to the positive property of damping, consisting of the decay of the natural oscillations of the mechanism, it must be remembered that, in a number of cases, this decay may also facilitate a certain increase in the dynamic error, since it produces a phase shift between vibrations of measurable force and the forced vibrations of the mechanism, and also since it affects the magnitude of the dynamic susceptibility.

In addition, in certain cases the damping itself may be a disturbing force which shifts the mechanism from its proper position (corresponding to the magnitude to be measured at the given moment) or, in other words, it is a source of dynamic error of Type II, which will be discussed in more detail in the next Section.

As indicated above, in many instruments used for measuring some force with the aid of an elastic element (specifically, in the majority of aircraft instruments of this type), dynamic errors of Type I are highly inconsequential. This is because the natural oscillation frequency of the mechanisms of such instruments is usually quite high, while the change in the magnitude of the forces to be measured takes place rather slowly in the majority of cases.

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Therefore, in developing the designs and calculations for instruments of this

type, dynamic errors of Type I are usually disregarded.

There are individual instruments, however, in which the natural oscillation frequency does not differ sufficiently from the possible frequency of change in the quantity to be measured (aircraft accelerographs are a typical example). In such cases, in addition to an analysis of the static errors in the instrument, it is necessary to make a more careful analysis of possible dynamic errors so as to find parameters yielding the lowest possible total error. In instrument design, allowance only for dynamic errors may lead to an increase in static errors, and allowance only for static errors may lead to the design of an instrument that is dynamically entirely worthless.

However, in view of the fact that the above statements apply only to individual instruments, and primarily in view of the fact that the question has been analyzed in detail in a number of well-known and widely-disseminated textbooks, we will give no further details here.

Section 6. Type II Dynamic Errors, Due to Acceleration of the Object on Which the Instrument Is Mounted

Magnitude of Perturbing Forces. A supplementary perturbing force $f_2(t)$ is usually the result of transferred accelerations in the instrument mechanism, i.e., those resulting from the presence of accelerations in the unit (ship, aircraft, etc.) to which the instrument is mounted, if the instrument contains unbalanced parts.

Note: In certain instruments, such as accelerometers and vibrometers, these accelerations are produced by the very force $f_1(t)$ being measured by the instrument.

It is quite obvious that these forces of inertia also require a reduction, by multiplication by the first power of the transmission ratio between the point in which the force of inertia develops and the point to which the forces are reduced. Therefore, under conditions in which, in the presence of linear transferred

accelerations of the entire instrument, it is desirable to write the reduced perturbing force $f_2(t)$ of eq.(2.4) immediately in the form of the product of some mass and the transferred acceleration, it must be remembered that in this case the reduced mass is generally not that contained in the inertia term producing acceleration on relative displacements of the mechanism relative to the instrument housing.

For example, at linear transferred accelerations and reduction of all forces to a linearly moving point, the "disturbing" mass is equal to the reduced force of inequilibrium of the mechanism in the direction opposite to that of the acceleration, divided by the acceleration of gravity g . In general, however, if we do not introduce this term for some "perturbing" force, the reduced perturbing force $f_2(t)$ is equal to the reduced force of inequilibrium of the mechanism in a direction opposite to that of acceleration, multiplied by the ratio $\frac{j}{g}$, where j is the transferred acceleration of the entire instrument.

The fact that the reduced perturbing force $f_2(t)$, resulting from linear transferred accelerations, is zero when all parts of the mechanism are balanced relative to their axes of rotation, is completely in agreement with the rule for reduction of the forces of inertia due to transferred accelerations via multiplication by the transmission ratio to the first power, since these transmission ratios from various points of the mechanism to the point to which all forces are reduced, may vary in sign. Nevertheless, in determining the reduced mass, we obtain uniform signs for all terms in the expression $m_1 \ddot{x}$ of eq.(2.8) when multiplying the mass of the individual material points in the mechanism by the square of the transmission ratio.

In a case where all forces and moments are reduced to an axis rather than to a point in linear motion, the reduced perturbation moment $M_2(t)$, resulting from the linear transferred acceleration j in the instrument, will obviously be equal to the reduced moment of unbalance of the mechanism, relative to the axis of reduction $STAT$ (which would result if the force of gravity acted in a direction opposite to the

acceleration), multiplied by the ratio $\frac{j}{g}$.

It is quite obvious that these remarks on the reduced forces of inertia and moments are entirely applicable to a case in which the forces and moments of inertia result from tangential and centripetal accelerations as the instrument housing rotates about the axis MN. Here, however, tangential and centripetal accelerations differ in magnitude and direction in various elements of the mechanism, depending on their position relative to the MN axis.

Here it is necessary to note the following: When an instrument is rotated in the manner indicated, the resultant tangential acceleration at any elementary material point of any part of the instrument can be expressed as the geometric sum of the tangential acceleration of the center of gravity of the part as it rotates about the axis MN, and the tangential acceleration resulting from the rotation about the axis M_1N_1 (Fig.11), passing through the center of gravity of the part and the parallel axis MN (the angular velocities relative to the axis M_1N_1 are obviously identical with those relative to MN).

Similarly, centripetal tendencies of any elementary material point of a part may be expressed as the geometric sum of the centripetal acceleration of the center of gravity of the part during its rotation about the MN axis and of the centripetal acceleration occurring during the rotation about the M_1N_1 axis (with the same velocity, of course, as about the MN axis).

Therefore, if any part of a mechanism is balanced about its axis of rotation (most parts in instrument mechanisms rotate about their axes), then in the general case only the accelerations of the center of gravity of the part (both tangential and centripetal) will actually fail to produce perturbation moments of inertia. As far as tangential and centripetal accelerations resulting from rotations of a part about the M_1N_1 axis are concerned, they may also cause perturbation moments when t_{STAT} part is balanced about its axis of rotation.

If the angle between the axis M_1N_1 and the axis Oz of rotation of the part is denoted by β , then in the presence of an angular acceleration $\dot{\omega}$, relative to the

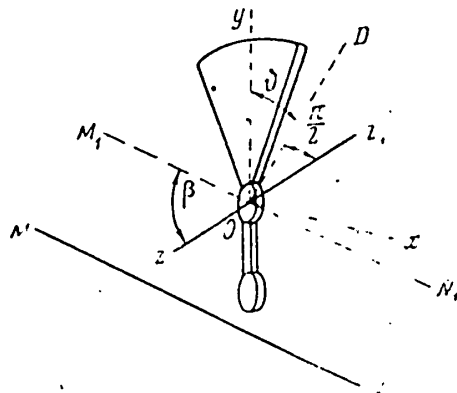


Fig.11 - Effect on Mechanisms of Angular Accelerations About an Axis MN

axis MN (and, therefore, about the axis M_1N_1), there is also an angular acceleration about the axis Oz equal to $\dot{\omega} \cos \beta$, resulting in a moment of inertia about the axis Oz equal to $J_z \dot{\omega} \cos \beta$, where J_z is the moment of inertia of the part relative to its axis of rotation Oz .

Likewise, as a result of the existence of a velocity ω of rotation about the MN axis (and there-

fore about the M_1N_1 axis), there exists an angular velocity equal to $\omega \sin \beta$ about some axis OD normal to the Oz axis of rotation of the part and lying in the same plane as the axes Oz and M_1N_1 . This velocity, $\omega \sin \beta$ creates a moment relative to the Oz axis, equal to

$$(J_{xz} - J_{yz}) \omega^2 \sin^2 \beta \sin \theta \cos \theta,$$

where J_{xz} and J_{yz} are the moments of inertia of the part relative to the two principal planes Ozx and Ozy passing through the axis Oz ;

θ is an angle made by the OD axis relative to any of the principal planes.

We will tentatively express the moment of inertia relative to the plane, by the sum $\sum_{i=0}^{i=n} (m_i a_i^2)$, where m_i is the mass of an elementary particle in the part, while a_i is the distance from the plane. The planes passing through the axis known as the principal axis are those relative to a plane whose moment of inertia is greatest,

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and relative to a plane whose moment of inertia is least. It is quite obvious that the two principal planes of inertia are always perpendicular to each other.

Note: The derivation of the formula for the moment $(J_{xz} - J_{yz})\omega_{OD}^2 \sin \theta \cos \theta$, generated about the axis Oz of rotation of the part, due to the angular velocity ω_{OD} of the rotation of this Oz axis about the OD axis normal thereto, is presented in a number of sufficiently well-known and widely-disseminated sources (for example, in the derivation of the formulas for centrifugal tachometers), and is therefore not given here.

Of course, the development of a moment due to centrifugal forces during rotation about the axis $M_1 N_1$ - this moment being represented by the expression $(J_{xz} - J_{yz})\omega^2 \sin^2 \theta \cos \theta \sin \theta$ - may easily be prevented by designing the part so that $J_{xz} = J_{yz}$, i.e., so that the moments of inertia of the part relative to all planes passing through the axis of rotation will be equal. However, there is no need to do this since these latter moments are usually so small as to be imperceptible.

With respect to the moments of inertia $-J_z \ddot{\omega}_z$ resulting from the angular accelerations $\ddot{\omega}_z$ about the axis of rotation of the part, it is quite obvious that it is impossible to prevent these by a proper design for a single part alone, since the moment of inertia of the part J_z is not converted to zero. However, in view of the fact that the forces and moments of inertia are obtained by multiplication by the transmission ratio to the first power, it is obvious that it is possible to compensate the disturbing moments of inertia, due to transferred angular accelerations, by a corresponding interaction of a number of parts.

For example, in a design in which two parts have parallel axes of rotation and rotate in opposite directions making the ratio of the absolute magnitudes of their velocities of rotation, relative to the body of the instrument, inversely proportional to their moments of inertia, the moments of inertia due to transferred angular velocities relative to their axes, will compensate each other. STAT

In fact, if we denote the total transferred velocity of rotation of the entire instrument by $\dot{\psi}$, the relative velocities of rotation of the two parts under examination by $\dot{\alpha}_1$ and $\dot{\alpha}_2$ (Fig. 12 illustrates a case in which the velocity $\dot{\alpha}_2$ is a negative quantity, while in our derivation we consider any motion in the direction of the velocities $\dot{\alpha}_1$ and $\dot{\psi}$ as positive), and if their moments of inertia are represented, respectively, by J_1 and J_2 , then the total reduced moment of inertia of the first part relative to the axis O_1 will become

$$\begin{aligned} M_r &= -J_1(\ddot{\psi} + \ddot{\alpha}_1) - J_2(\ddot{\psi} + \ddot{\alpha}_2) \frac{d\alpha_2}{d\alpha_1} = \\ &= -\ddot{\psi} \left(J_1 + J_2 \frac{d\alpha_2}{d\alpha_1} \right) - J_1 \ddot{\alpha}_1 - J_2 \ddot{\alpha}_2 \frac{d\alpha_2}{d\alpha_1}. \end{aligned}$$

And since, in the general case,

$$\ddot{\alpha}_2 = \ddot{\alpha}_1 \frac{d\alpha_2}{d\alpha_1} + \dot{\alpha}_1^2 \frac{d^2\alpha_2}{d\alpha_1^2},$$

then

$$\begin{aligned} M_r &= -\ddot{\psi} \left(J_1 + J_2 \frac{d\alpha_2}{d\alpha_1} \right) - \left[J_1 + J_2 \left(\frac{d\alpha_2}{d\alpha_1} \right)^2 \right] \ddot{\alpha}_1 - \\ &\quad - J_2 \left(\frac{d^2\alpha_2}{d\alpha_1^2} \right) \left(\frac{d\alpha_2}{d\alpha_1} \right) \dot{\alpha}_1^2. \end{aligned} \quad (2.20)$$

Since the expression $\frac{d\alpha_2}{d\alpha_1}$ is negative and $\left| \frac{d\alpha_2}{d\alpha_1} \right| = \frac{J_1}{J_2}$, then the coefficient is zero at transferred angular acceleration, while the reduced moments of inertia at a relative acceleration $\ddot{\alpha}_1$ always total to a clearly positive magnitude, since the STAT value of J_2 is multiplied by the square of the $\frac{d\alpha_2}{d\alpha_1}$ ratio. However, if the angular transferred accelerations are to be compensated at all positions of the parts in

question relative to each other, then the condition $\left| \frac{d\alpha_2}{d\alpha_1} \right| = \frac{J_1}{J_2}$ must be maintained in all these positions, or in other words the $\frac{d\alpha_2}{d\alpha_1}$ ratio must be constant. In this case, the last item in expression (2.20), i.e., the term containing the $\dot{\alpha}_1^2$ factor, is converted to zero, and M_r turns out to be

$$M_r = - \left[J_1 + J_2 \left(\frac{d\alpha_2}{d\alpha_1} \right)^2 \right] \ddot{\alpha}_1.$$

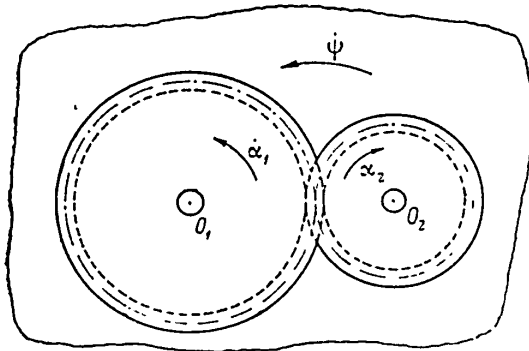


Fig.12 - Possibility of Compensating the Effect of Angular Accelerations Resulting from the Interaction of Several Parts

It should be noted, however, that only in rare cases is compensation of transferred angular accelerations provided, even in designs for instruments meant to operate under conditions of high velocities and accelerations (as in an aircraft), since this results in major design complications, compared to which angular accelerations in the functioning of most instruments are very small and of short duration. Therefore, efforts are limited to balancing parts around their axes of rotation, i.e., to compensation solely of linear transferred accelerations. This is not complex. Moreover, in the practical use of instruments, the accelerations involved are usually large, linear, and persistent (including accelerations from rotation about a large radius, as for example centripetal accelerations in aircraft turns).

In addition to the basic difference, indicated above, between the method of

reducing the inertia forces resulting from transferred accelerations, and the method of determining the reduced mass in terms of an equation for the motion of a mechanism due to accelerations in its relative displacements, there are possibly other

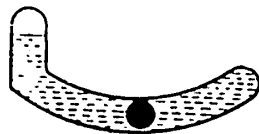


Fig.13 - Turn-and-Bank Indicator
for Aircraft

causes; these have the effect that the mass contained in the term for the mechanism's equation of motion, which results from transferred accelerations, differs from the reduced mass contained in the term for accelerations due to relative displacements of the mechanism.

Let us take, as a typical example, a case in which the mechanism is placed in a liquid, filling the body of the instrument. Such a device is sometimes used for attenuating the natural vibrations of the mechanism. As an example we may cite the turn-and-bank indicator of an aircraft (Fig.13). The instrument consists of a glass tube shaped as an arc of a circle, within which a metal ball is free to roll. The tube is filled with liquid to damp the vibrations of the ball. In the presence of a transferred linear acceleration j , sensed by the body of such an instrument, the transfer of an equal acceleration j to a part of a mechanism of mass m_0 is related to the development of a force of inertia of $(m_0 - m_1)j$, where m_1 is the mass of an amount of liquid equal in volume to the volume of the part. This is the only force that has to be considered in compiling the general equation, because of the fact that the additional force $m_1 j$ which, added to the force $(m_0 - m_1)j$ yields the total force $m_0 j$, required for the part to attain an acceleration of j , results from the pressure of the liquid on the part, due to accelerations of the body of the instrument. However, the total mass m_0 of the part must be written into that term in the equation of motion of the mechanism that provides for acceleration relative to the motion of the mechanism.

STAT

Note: Here we are not speaking of cases in which individual portions of the part in question are not homogeneous in specific gravity, i.e., cases where the center of the volume of the part does not coincide with the center of its mass. In this case, obviously, accelerations of the body of the instrument produce an additional perturbing moment, i.e., a couple equal to $m_1 j a$, where a is the projection of the distance between the center of mass and the center of volume in a plane normal to the direction of acceleration j . In reality, in order for the part to acquire an acceleration j , it is necessary to apply a force $m_0 j$ to the center of the mass of the part (with due consideration for the force of pressure of the liquid $m_1 j$, resulting in the presence of an acceleration j and applied to the center of volume of the part and a force $m_1 j$ opposite in direction to its center of volume.

This case of a mechanism placed in a liquid also corresponds fully to the general formulation which states that the reduced perturbing force (or moment) resulting from linear transferred accelerations of the entire instrument is equal to the reduced force (or moment) of inequilibrium of the instrument which would result if the force of gravity were in a direction opposite to the acceleration, multiplied by $\frac{j}{g}$.

By way of illustration let us here derive the equation of motion of the ball of a turn-and-bank indicator and the equation of motion of the air bubble in a liquid level.

To determine the nature of motion of the turn-and-bank indicator ball, we may use an equation analogous to that for a physical pendulum (with caging mechanism), taking the center of curvature of the turn-and-bank indicator arc tube as the point of suspension of the pendulum. STAT

The equation for the motion of a physical pendulum with arrester in the presence of vertical acceleration j_z and horizontal acceleration j_x of the point of

suspension (counting the acceleration j_z as positive in the upward direction, and j_x as positive leftward), shown in Fig.14, has the following form:

$$J\ddot{\varphi} + B(\dot{\varphi} - \dot{\beta}) + Pl \sin \varphi - M_r = lm(j_x \cos \varphi - j_z \sin \varphi). \quad (2.21)$$

where φ is the angle of deviation of the pendulum from the vertical;

J is the moment of inertia of the pendulum;

B is the damping factor (for reasons of simplicity we choose a case of damping proportional to the velocity to the first power);

$\dot{\beta}$ is the angular velocity of rotation of the housing to which the arrester is attached, in motion about the axis O of the pendulum (in this case, the damping will obviously be proportional to the relative velocity $\dot{\varphi} - \dot{\beta}$);

P is the weight of the pendulum;

l is the distance of its center of gravity from the point of suspension;

m is the mass of the pendulum;

M_{fr} is the moment of friction in the axis O , which we will consider as constant in absolute magnitude.

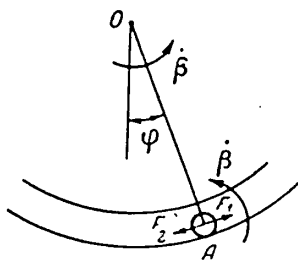


Fig.14 - Derivation of Moment of Inertia Due to Transferred Acceleration $\ddot{\beta}$ for an Inclinator Rotating about the Axis O

axis O , while the angle $\varphi - \beta$ is the angular relative displacement of the center of

It is necessary to modify the equation for the turn-and-bank indicator. The term $J\ddot{\varphi}$ has to be replaced by another expression determined from the following considerations: Let us divide the acceleration $\ddot{\varphi}$ into two components: $\ddot{\beta}$ and $\ddot{\varphi} - \ddot{\beta}$, where the angle β describes the transferred displacement of the ball with the tube as the latter rotates about

the ball relative to the tube. In this case, we assume that the instrument will, in addition to the linear displacements of the point O with the accelerations j_z and j_x , also present certain rotations about the axis O, disregarding for the time being the rolling of the ball along the tube.

In the presence of an angular acceleration, $\ddot{\beta}$, a linear acceleration of $\ell \ddot{\beta}$ takes place at the center of the ball, where ℓ is the radius of curvature of an arc along which the center of the ball can be displaced in the tube. In order for the ball to acquire the indicated acceleration, a force equal to $m_0 \ell \ddot{\beta}$ is required, where m_0 is the mass of the ball. However, as already indicated, a portion of this force, equivalent to $m_1 \ell \ddot{\beta}$ (where m_1 is the mass of a volume of liquid equal to the volume of the ball) was obtained as a result of the pressure of the liquid, produced by the acceleration $\ell \ddot{\beta}$. Therefore, in the equation of motion of the ball, it is necessary to allow only for the force of inertia $-(m_0 - m_1) \ell \ddot{\beta}$, creating, relative to the axis O, a moment of inertia equal to $-(m_0 - m_1) \ell^2 \ddot{\beta}$.

The ball, moving with the tube around the axis O (see Fig.14) rotates about its diameter parallel to the axis O, with the same velocity and in the same direction as the tube rotates about the O axis (disregarding, for the time being, the rolling of the ball along the tube). In rotations with an acceleration $\ddot{\beta}$, a moment of $J_1 \ddot{\beta}$ is required, where J_1 is the moment of inertia of the ball about its diameter. The moment of inertia developing in this fashion, having a sign opposite to that of the moment required for producing the acceleration, is opposite to that shown by the arrow in Fig.14.

The point of support of the ball is at the point A, and the acceleration $\ddot{\beta}$ requires a force F_2 , so that we obtain a force of inertia F_1 , passing through the center of the ball and equal to

$$F_1 = \frac{J_1 \ddot{\beta}}{r}$$

where r is the radius of the ball.

STAT

This force of inertia F_1 creates a moment of inertia about the axis 0. This moment is equal to $+J_1 \frac{l}{r} \ddot{\beta}$.

Thus, the presence in the ball of a transferred angular acceleration $\ddot{\beta}$ about the axis 0 produces a moment of inertia $(M_1)\ddot{\beta}$, relative to the axis 0, which equals

$$(M_1)\ddot{\beta} = -\left[(m_0 - m_1)l^2 - J_1 \frac{l}{r}\right] \ddot{\beta}. \quad (2.22)$$

In the presence of a relative acceleration $l(\ddot{\varphi} - \ddot{\beta})$, a moment of inertia relative to the axis 0 equal to $-m_0 l^2 (\ddot{\varphi} - \ddot{\beta})$ appears at the center of the ball.

This being the case, the ball rolls along the tube (assuming, for purposes of simplicity, that the ball does not slip) rotating about its diameter at a velocity opposite in direction to that of the velocity $\dot{\varphi}$ (Fig. 15). The magnitude of the velocity $\dot{\theta}$ has a specific relation to the magnitude $\dot{\varphi} - \dot{\beta}$, which may be determined as follows: At the point A of the ball, which, at the given moment of time, is in contact with the tube, the linear velocity relative to the tube is zero. In this case, the velocity in question is the algebraic sum of the velocity $l(\dot{\varphi} - \dot{\beta})$ of the center of the ball and the tangential velocity $r\dot{\theta}$ of the point A of the ball, as it rotates with a velocity of $\dot{\theta}$. Thus,

$$l(\dot{\varphi} - \dot{\beta}) = -r\dot{\theta},$$

from which we obtain

$$\dot{\theta} = -\frac{l}{r} (\dot{\varphi} - \dot{\beta})$$

and

$$\ddot{\theta} = -\frac{l}{2} (\ddot{\varphi} - \ddot{\beta}).$$

Rotation at an angular acceleration of $\ddot{\theta}$ requires the application of a moment relative to the diameter of the ball, the absolute magnitude of this moment

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being $J_1 \frac{l}{r} (\ddot{\varphi} - \ddot{\beta})$.

This requires that a force F_i , equal to $J_1 \frac{l}{r_2} (\ddot{\varphi} - \ddot{\beta})$, be applied to the center of the ball. In other words, it requires a moment relative to the axis O that would be equal to $+J_1 (\frac{l}{r})^2 (\ddot{\varphi} - \ddot{\beta})$. Thus, the moment of inertia relative to the axis O, arising as a result of the acceleration $\ddot{\theta}$ as the ball rolls about the tube, is equal to $-J_1 (\frac{l}{r})^2 (\ddot{\varphi} - \ddot{\beta})$.

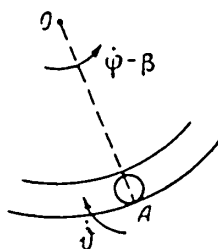


Fig.15 - Derivation of an Expression for the Moment of Inertia Due to Relative Acceleration $\ddot{\varphi} - \ddot{\beta}$, as the Ball Is Displaced Along Its Tube

In addition, it is also necessary to take into consideration the reduced moment due to accelerations during displacement of the liquid as the ball rolls through the tube at an acceleration of $\ddot{\varphi} - \ddot{\beta}$. This moment is a function $\phi(\ddot{\varphi} - \ddot{\beta})$ of the acceleration $(\ddot{\varphi} - \ddot{\beta})$. It should be noted that the form and magnitude of the moment $\phi(\ddot{\varphi} - \ddot{\beta})$ are very difficult to determine theoretically, since

it is impossible to establish exactly which particles of the liquid actually move and at what speed this motion takes place (while, as we know, in determining the reduced moments of inertia derived from relative accelerations as a function of the acceleration $(\ddot{\varphi} - \ddot{\beta})$ about the axis of reduction, it is necessary to multiply the reduced mass by the square of the ratio of the velocity of displacement of this mass to the rate $\dot{\varphi} - \dot{\beta}$ of displacement of the ball). STAT

Thus, the presence of a relative angular acceleration about the axis O requires

the development of a moment of inertia $(M_{\bar{n}})\ddot{\varphi}-\ddot{\beta}$ relative to the axis 0, equal to

$$(M)\ddot{\varphi}-\ddot{\beta} = -\left[m_0 l^2 + J_1 \left(\frac{l}{r}\right)^2\right](\ddot{\varphi}-\ddot{\beta}) + \Phi(\dot{\varphi}-\dot{\beta}). \quad (2.23)$$

Thus, in place of member $J_{\bar{\varphi}}$ in eq.(2.21) we now have

$$\begin{aligned} & \left[(m_0 - m_1)l^2 - J_1 \frac{l}{r}\right] \ddot{\beta} + \\ & + \left[m_0 l^2 + J_1 \left(\frac{l}{r}\right)^2\right](\ddot{\varphi}-\ddot{\beta}) + \Phi(\dot{\varphi}-\dot{\beta}). \end{aligned} \quad (2.24)$$

In the terms $Pl \sin \varphi = m_0 l \sin \varphi$ and $ml(j_x \cos \varphi - j_z \sin \varphi)$ it is also necessary to substitute the quantity m by the quantity $m_0 - m_1$ rather than by the total mass of the ball m_0 .

The resultant expressions for moments of inertia in the equation of motion of the turn-and-bank indicator are entirely in accord with the above remarks. In fact, the ratio of the rate of rotation of the ball about its diameter as it rolls in the tube, to the resultant rate of rotation of the straight line connecting the center of the ball with the axis 0 is equal to $-\frac{l}{r}$ (the minus sign being due to the fact that the two velocities of rotation are opposite in sign). In this connection, in the term resulting from the relative acceleration $\ddot{\varphi}-\ddot{\beta}$, the reduced moment of inertia of the ball is equal to $J_1 \left(-\frac{l}{r}\right)^2 = J_1 \left(\frac{l}{r}\right)^2$. However, in the reduced moment of inertia due to the transferred acceleration $\ddot{\beta}$, we obtain instead

$$J_1 \left(-\frac{l}{r}\right) = -J_1 \frac{l}{r}. \quad \text{STAT}$$

In addition, in the moments of inertia $(m_0 - m_1)l^2 \ddot{\beta}$ and $(m_0 - m_1)(j_x \cos \varphi - j_z \sin \varphi)$,

resulting from the linear transferred accelerations, we assume a mass $m_0 - m_1$, and in the moment of inertia $m_0 l^2 (\ddot{\varphi} - \ddot{\beta})$, due to relative acceleration, a mass m_0 .

The reasoning is identical, for the bubble level. However, the terms $J_1 \frac{l}{r} \ddot{\beta}$ and $J_1 (\frac{l}{r})^2 (\ddot{\varphi} - \ddot{\beta})$, resulting from the rotation of the ball about its diameter, do not have to be written, since, for the case of the bubble level, these terms are preceded by a sign opposite to that used in the equation for the ball-type inclinometer. This has to be done since the glass in the bubble level is convex in the opposite direction (upward) from that of the ball-type inclinometer. Therefore the numerical values of these terms do not change in sign as a result.

Thus, for the bubble level, eq.(2.21) is replaced by the equation

$$-(m_1 - m_0) l^2 \ddot{\beta} + m_0 l^2 (\ddot{\varphi} - \ddot{\beta}) + \Phi(\varphi - \beta) + B(\dot{\varphi} - \dot{\beta}) + (m_1 - m_0) l (j_z \sin \varphi - j_x \cos \varphi) \pm M_{tp} = -(m_1 - m_0) g l \sin \varphi \quad (2.25)$$

while for the ball-type inclinometer we will have

$$\begin{aligned} & \left[(m_0 - m_1) l^2 - J_1 \frac{l}{r} \right] \ddot{\beta} + \left[m_0 l^2 + J_1 \left(\frac{l}{r} \right)^2 \right] (\ddot{\varphi} - \ddot{\beta}) + \Phi(\varphi - \beta) + \\ & - B(\dot{\varphi} - \dot{\beta}) + (m_0 - m_1) l (j_z \sin \varphi - j_x \cos \varphi) \pm M_{tp} = \\ & = -(m_0 - m_1) g l \sin \varphi. \end{aligned} \quad (2.26)$$

A comparison of these two levels shows that if they are solved relative to the angle $\varphi - \beta$, i.e., if the displacement of the ball relative to the tube or bubble is determined in terms of the level chamber, the acceleration $\ddot{\beta}$ creates, in the bubble, a perturbing force acting in a direction opposite to that of the action of the same perturbing force if a ball-type inclinometer. The latter is a result of the fact that, for a ball-type inclinometer, the expression $(m_0 - m_1) l^2 - J_1 \frac{l}{r}$ is clearly positive. STAT

This is quite understandable from the physics viewpoint since, in a bubble level, the existence of the acceleration $\ddot{\beta}$ will cause the liquid to tend (because of its inertia) to occupy the rear portion of the tube (looking in the direction of motion) and, therefore, to push the bubble forward. With a ball we have the opposite phenomenon, since the ball, having a higher specific gravity than the liquid, has greater inertia. In a spherical bubble level, the quantity m_0 is so small (considerably smaller than the magnitude m_1) that it can be neglected.

In this case, the quantity $\Phi(\ddot{\varphi}-\ddot{\beta})$ is also not great. As a result, bubble levels usually have very small periods of natural vibration and thus smaller dynamic errors than, for example, the ball-type inclinometers described above, if they are used as turn-and-bank indicators, i.e., as instruments showing the direction of the so-called "apparent" vertical. But this indicates that they are less reliable as indicators of the true vertical when installed in aircraft (although they are used in sextants, gunsights, etc.), since accelerations of some kind are always present in an aircraft.

Section 7. Type II Dynamic Errors Due to Damping Action in Instruments

Indicating Direction Relative to the Earth's Surface or To Space

As indicated above, dampers may sometimes be sources of disturbances in mechanisms, i.e., they may cause mechanisms to deviate from their correct positions. It is true that this pertains primarily to instruments in which the magnitude of the position force is a function of the displacement of the mechanism or any of its parts from some direction relative to the earth's surface or to a stationary space, regardless of the position of the body of the instrument.

All statements in the preceding paragraphs pertain not only to instruments in which the magnitude of the position force (or moment) is a function of the deviation of the mechanism relative to the body of the instrument, but also to instruments for indicating direction due to the presence of some force acting in the given direction (such as pendulums, compasses, etc.).

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Let us note also that in analyses of these instruments, the factor φ in eq.(2.8), in the majority of cases, refers not to the magnitude of the relative deviation of the mechanism with respect to the body of the instrument (which is usually covered by the quantity x or φ in instruments measuring some force by means of a spring) but to the magnitude of deviation of the given element of the mechanism from the given direction. In this case, the force $f_1(t)$ does not appear in the right side of eq.(2.8), so that the entire quantity φ is an error in the instrument readings (only dynamic errors of the second order being present). The readings of the instrument obtained as a result of deviations of the mechanism relative to the body of the instrument are in this case determined by the angle $\beta - \varphi$, where β is the angle of deviation of the instrument mount from the given direction, i.e., the angle β corresponds to a correct instrument reading.

In instruments of the latter type, the arrester may often be the source of a considerable perturbing force. In general, it should be noted that instruments of this category, under certain conditions, are subject to excessive perturbing forces, while instruments measuring a force with the aid of an elastic element are usually subject to considerably weaker perturbing forces.

By way of an example of the interfering action of an arrester, let us examine the behavior of a pendulum-type inclinometer. Let us assume that the aircraft pitches fore and aft (as is usually the case) about an axis parallel to the axis O , on which the pendulum rocks. If the O axis of the pendulum does not coincide with the axis of pitch of the aircraft, accelerations of considerable magnitude will appear at the point O in the aircraft suspension, since pitching (i.e., periodic rotation first in one and then in the other direction) cannot occur without angular accelerations $\ddot{\beta}$, as a result of which the point of suspension of the pendulum will acquire periodic accelerations of $r \ddot{\beta}$ (where r is the distance of the O axis from the aircraft axis of pitch). In addition, centripetal accelerations are generated under this condition.

Even if it is assumed, however, that the 0 axis coincides with the axis of rotation of the aircraft, and the moment of friction M_{fr} is disregarded, i.e., if it is assumed that the vibrations of an aircraft do not contain a disturbing moment and if, finally, the inclinometer is considered as a simple damping physical pendulum, we have the equation

$$J\ddot{\varphi} + B(\dot{\varphi} - \dot{\beta}) + Pl \sin \varphi = 0$$

where the symbols are the same as in eq.(2.21).

For small vibrations of the instrument we may write

$$J\ddot{\varphi} + B\dot{\varphi} + Pl\varphi = B\dot{\beta}. \quad (2.27)$$

In this equation, the term $B\dot{\beta}$ is a perturbation term. This is due to the fact that, in the usual arresters, the damping takes place relative to the point of installation of the arrester, which itself moves with the body of the instrument, thus inducing forces tending to force the pendulum out of position (Fig.16). Let us examine some examples of the effect of such forces.

Let us assume that the aircraft rocks harmonically with an angular velocity ω , in radians per second, and with an amplitude of β_0 , i.e., that its deviation from the center position is

$$\beta = \beta_0 \cos \omega t.$$

Then,

$$\dot{\beta} = -\beta_0 \omega \sin \omega t,$$

giving the solution of eq.(2.27) in the following form:

$$\varphi = -\frac{\beta_0 B \omega}{J \sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} \sin(\omega t - \delta) + e^{-ht} (c_1 \sin \sqrt{k^2 - h^2} t + c_2 \cos \sqrt{k^2 - h^2} t =$$

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$$= -\beta_0 \frac{2h\omega}{\sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} \sin(\omega t - \delta) + e^{-ht} (c_1 \sin \sqrt{k^2 - h^2} t + c_2 \cos \sqrt{k^2 - h^2} t). \quad (2.28)$$

(The significance of k and h is the same as in the Chapter on dynamic errors).

Thus, even in the case of steady-state vibrations of an aircraft, the pendulum will still be under forced vibrations with the amplitude

$$\beta_0 \frac{2h\omega}{\sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}}.$$

In the case of, for example, 50% damping, we will have the amplitude

$$\varphi_0 = \beta_0 \frac{k\omega}{\sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} = \beta_0 \frac{1}{\sqrt{\left(\frac{k^2 - \omega^2}{k\omega}\right)^2 + 1}},$$

and, in a case in which $h^2 = 0.5k^2$ (75% damping), the amplitude will be:

$$\begin{aligned} \varphi_0 &= \beta_0 \frac{k\omega}{\sqrt{\frac{(k^2 - \omega^2)^2 + 2\omega^2 k^2}{2}}} = \beta_0 \frac{k\omega}{\sqrt{\frac{k^4 + \omega^4}{2}}} = \\ &= \beta_0 \frac{1}{\sqrt{\left(\frac{k^2}{\omega^2} + \frac{\omega^2}{k^2}\right) \frac{1}{2}}}. \end{aligned}$$

From this it is clear that, in order for the pendulum not to be taken too far from the vertical position by the arrester, it is necessary that its natural frequency k be as large as possible relative to the possible frequency ω of vibration of the aircraft. This requirement must be made since it is very difficult to make

a pendulum of k frequency significantly smaller than the vibration frequency ω of the aircraft.

Thus, at $k = \omega$, we have an amplitude φ_0 of forced vibrations of the pendulum equal to the amplitude β_0 of oscillations of the aircraft, at any damping not lagging by 90° .

In a case in which $k = 2\omega$, we have the following amplitudes:

at 50% damping:

$$\varphi_0 = \beta_0 \frac{1}{\sqrt{\left(2 - \frac{1}{2}\right)^2 + 1}} = \beta_0 \frac{1}{\sqrt{3,25}} \approx 0,5555 \beta_0;$$

at 70% damping:

$$\varphi_0 = \beta_0 \frac{1}{\sqrt{\frac{1}{2} \left(4 + \frac{1}{4}\right)}} = \beta_0 \frac{1}{\sqrt{2,125}} \approx 0,686 \beta_0.$$

In a case in which $k = 4\omega$, the amplitudes will be:

at 50% damping:

$$\varphi_0 = \beta_0 \frac{1}{\sqrt{\left(4 - \frac{1}{4}\right)^2 + 1}} = \beta_0 \frac{1}{\sqrt{\frac{241}{16}}} \approx 0,2577 \beta_0;$$

at 70% damping:

$$\varphi_0 = \beta_0 \frac{1}{\sqrt{\frac{1}{2} \left(16 + \frac{1}{16}\right)}} = \beta_0 \frac{1}{\sqrt{\frac{257}{32}}} \approx 0,3294 \beta_0.$$

Consequently, even when $k = 4\omega$, significant vibrations of the pendulum due to the arrester will occur, since the amplitude β_0 of aircraft vibrations may easily reach 3 to 5° . However, any significant reduction in the degree of shock absorption

leads to a slow decay of the natural vibrations of the pendulum.

Therefore, it is most desirable to use pendulums with the greatest possible

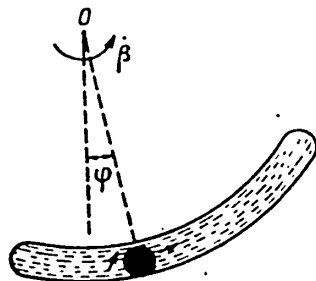


Fig.16 - Entrainment of a Pendulum by a Damper as the Point of Installation of the Pendulum Rotates

natural frequency, with respect to the frequency of vibration of the aircraft. This is advantageous also with respect to accelerations since, in this case, it is desirable that the natural frequency of the pendulum is as much as possible larger than the frequency of the accelerations acting on the aircraft. In that case, the discrepancy between the positions of the pendulum corresponding to its forced motion, and its

"static" position will be that much smaller.

Thus, when this condition is satisfied, the pendulum will always become set more accurately (with less lag) in the direction of the equivalent force of gravity and in the direction of all the forces of inertia created by the accelerations of an aircraft. Actually, this is the only requirement that this type of inclinometer is expected to meet.

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CHAPTER III

EFFECTS OF VIBRATIONS ON INSTRUMENT READINGS

Section 1. Forced Oscillations of Mechanisms During Vibrations

Any vibration of the mounting of an instrument is transmitted to its mechanism in the form of periodic, usually small, displacements of the body of the instrument from one side to the other, and in the form of associated displacements of periodic transferred accelerations of the entire mechanism, often of considerable magnitudes. Thus, the effect of vibration, as it were, is a special case of the problem examined above, of the effect of transferred accelerations on instrument mechanisms. Nevertheless, the effect of vibrations should be given special consideration, in view of the fact that efforts to combat such vibrations are often among the major problems to be solved in instrument design.

In reality, without speaking of the fact that vibration as such has a most destructive effect on many instruments, it is capable of inducing significant relative vibrations in the mechanisms. This complicates and sometimes even renders impossible a reading of the instrument. The most common factor causing indications to be unreadable is that the instrument mechanism enters into resonance with the vibrations, i.e., the frequency of the natural oscillations of the mechanism and the frequency of the vibrations become equal or very nearly so. Let us add further that, in many instruments designed for use under conditions in which their mountings vibrate, the frequency of the natural oscillations is of the same order as the frequencies at which it is possible for oscillations to occur. Therefore, if the

designer, striving to improve certain positive properties of the instrument (i.e., to increase the reading accuracy) does not allow for the disturbing effects of vibrations, he risks creating an unworkable instrument.

It was mentioned above that the dynamic analysis of many instruments is limited to a calculation of the frequency of the natural oscillations of the mechanism, with an effort to ensure that this frequency will differ to a maximum from the possible frequencies of vibration of the place of installation of the instrument since, in many instruments, the other dynamic errors are usually small. Naturally, the balancing of the parts of a mechanism on their axes of rotation for the purpose of compensating the effect of transferred accelerations in the instrument is not considered a part of dynamic analysis, although this struggle with the effect of accelerations is also a struggle to overcome dynamic errors in the instrument.

Let us examine the effect of vibrations, distinct from other dynamic errors in the instrument. For this, let us examine the effect of vibrations when the force measured by the instrument is a steady-state phenomenon of constant magnitude, with the relative displacements of the mechanism resulting only from vibration effects. In this case, the motion of the mechanism relative to the body of the instrument (disregarding the force of friction) can be described by the equation

$$m\ddot{x} + B\dot{x} + Sx + F_{\text{mean}} = F - \frac{P_{\text{ineq}}}{g} j, \quad (3.1)$$

where P_{ineq} is the reduced force of inequilibrium that would result if the force of gravity had the same direction as the vibrations;

j is the acceleration in the housing of the instrument during vibration and is a periodic function;

F is the force being measured, which we consider a constant; STAT

x is the displacement first to one side and then to the other due to vibration, away from a midpoint position of the reduced total forces

in the mechanism;

F_{mean} is the magnitude of the position force when the mechanism is in this center position. The remaining symbols have been defined above.

In the absence of vibration, eq.(3.1) is simply replaced by the equation $F_p = F$, where F_p is the magnitude of the position force in the absence of vibration.

Here, it must be remembered that, strictly speaking, the force F_p may, in particular cases, differ somewhat from the force of F_{mean} during vibration. In other words, when vibration is present, the difference $F_{\text{mean}} - F$ in eq.(3.1) may sometimes differ slightly from zero. This obviously involves some change in the averaged readings of the instrument during vibration. This is because due to the fact that the force $\frac{P_{\text{ineq.}}}{g} j$, periodically changing its direction, may in certain cases act more strongly in one direction than in another, due to the possible inconstancy of the magnitude of $P_{\text{ineq.}}$ even in purely periodic harmonic vibration. This will be discussed in somewhat greater detail below, in a Section devoted to various causes of the changes in mean instrument readings, sometimes noted in the presence of vibrations. As far as the situation is concerned in which, with oscillations of the mechanism as a whole, the quantity F_{mean} differs slightly from the quantity F_p , then, as has been shown by theoretical studies and by practice, this difference is imperceptibly small in the instruments under discussion, since the magnitude of excess action by the force $\frac{P_{\text{ineq.}}}{g} j$ in one direction is very small relative to its effect in the other.

In view of this fact, in investigating the question as to the magnitude of the amplitude of oscillations of vibrating mechanisms, it is entirely permissible in the majority of cases to regard the quantity $P_{\text{ineq.}}$ as a constant; and therefore to consider the quantity F_{mean} as equal to the quantity F_p and thus to F . As a result, we obtain an equation of the form

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$$m\ddot{x} + B\dot{x} + Sx = -\frac{P}{g}j. \quad (3.2)$$

Let us illustrate this with the simplest case: that in which the vibrations are harmonic.

If we denote by a the amplitude of oscillation of the instrument housing in the presence of vibration and by ω the angular velocity of vibration, then the displacement of the instrument y will coincide with the following relation:

$$y = a \sin(\omega t + \varepsilon),$$

where ε is a constant.

It is obvious that, with this type of motion, the acceleration will be

$$\ddot{y} = -a \omega^2 \sin(\omega t + \varepsilon).$$

Then the perturbing force in eq.(3.2) (term $-\frac{P_{ineq.}}{g} j$) will take the form

$$-\frac{P_{ineq.}}{g} j = a \frac{P_{ineq.}}{g} \omega^2 \sin(\omega t + \varepsilon_1).$$

If this perturbing force is substituted in eq.(3.2), the forced relative motion of the point to which the forces are reduced (relative to the instrument housing) will have the following form [see eq.(2.16)]:

$$\begin{aligned} x_{forced} &= a \frac{P_{ineq.}}{gm} \frac{\omega^2}{\sqrt{(k^2 - \omega^2)^2 + 4\omega^2 h^2}} \sin(\omega t + \varepsilon - \delta) = \\ &= a \frac{P_{ineq.}}{gm} \frac{1}{\sqrt{\left(\frac{k^2}{\omega^2} - 1\right)^2 + 4 \frac{k^2}{\omega^2} \frac{h^2}{k^2}}} \sin(\omega t + \varepsilon - \delta), \end{aligned} \quad (3.3)$$

where δ is the phase shift equal to

$$\delta = \arctg \frac{2\omega h}{k^2 - \omega^2}.$$

Thus, the amplitude of the relative oscillations of the point is proportional STAT to the vibration amplitude a , and to the quantity $\frac{P_{ineq.}}{gm}$, where m is the total reduced mass of the mechanism, as well as to the expression

$$\frac{1}{\sqrt{\left(\frac{k^2}{\omega^2} - 1\right)^2 + 4 \frac{k^2}{\omega^2} \frac{h^2}{k^2}}}$$

This last expression, which will be denoted by λ_1 , is a function of the ratio $\frac{\omega}{k}$ and the damping coefficient $\frac{h}{k}$.

Figure 17 shows curves for the value of λ_1 , depending on the magnitude of the $\frac{\omega}{k}$ ratio for various degrees of damping. As might have been expected (on the basis of the curves in Fig. 10), the reduced curves, starting at the value for the degree of damping at which $h^2 = 0.5k^2$, display no maximum within finite values of $\frac{\omega}{k} > 0$ and the quantity λ_1 , nor may the $\frac{\omega}{k}$ ratio be greater than unity at any values whatever. In reality then, in the given case, the minimum of the quantity N below the square root sign, which is

$$N = \left(\frac{k^2}{\omega^2} - 1\right)^2 + 4 \frac{k^2}{\omega^2} \frac{h^2}{k^2},$$

can be obtained from the equation

$$\frac{-4\left(\frac{k^2}{\omega^2} - 1\right) - 8 \frac{h^2}{k^2}}{\frac{\omega^3}{k^3}} = -4 \frac{k^3}{\omega^3} \left(\frac{k^2}{\omega^2} - 1 + 2 \frac{h^2}{k^2}\right) = 0.$$

Obviously, the expression in parentheses can be equal to zero only until $2 \frac{h^2}{k^2} < 1$, and in this case we will have a minimum, since

$$\frac{d^2 N}{d\left(\frac{\omega}{k}\right)^2} = 12 \frac{k^4}{\omega^4} \left(\frac{k^2}{\omega^2} - 1 + 2 \frac{h^2}{k^2}\right) + 8 \frac{k^6}{\omega^6}.$$

However, since

STAT

$$\frac{k^2}{\omega^2} - 1 + 2 \frac{h^2}{k^2} = 0,$$

it follows that

$$d \left(\frac{\omega}{k} \right)^2 = 8 \frac{k^6}{\omega^6},$$

which is thus a positive quantity.

Thus, at damping in the 70% range, the vibrations do not increase even in the presence of resonance. When the $\frac{\omega}{k}$ ratio is larger, the quantity λ_1 is close to unity. In that case, the phase shift is close to 180° .

There is no special provision for shock absorption in many instruments, and their degree of damping is therefore not large. Equation (3.3) shows that, in designing instruments, the parameters of the mechanism must be arranged so that the $\frac{k}{\omega}$ ratio is not close to unity. The factor λ_1 facilitates a reduction in the amplitude of the relative forced oscillations of the mechanism due to vibration, in a case in which $k > \omega$ (of course, to an adequate degree). The amplitude of these relative oscillations of the mechanism declines with an increase in the $\frac{k}{\omega}$ ratio.

However, in many instruments where $k < \omega$, so that λ_1 is close to unity (or even somewhat greater than unity), the amplitudes of the forced relative oscillations due to vibration are only a fraction of the vibration amplitude a . This is explained by the fact that, in most instruments, the term mg in eq.(3.3) where m is the reduced mass of the mechanism in relative displacement, is many times greater than the quantity P_{ineq} . since, in instruments designed to function under conditions of vibration, the parts of the mechanism are all balanced, (if possible), relative to their axes of rotation. Therefore, the effort of counteracting the vibrations effects on the instrument readings usually are reduced to making the frequencies k and ω sufficiently different.

In instruments designed for measuring vibrations, the design is such as to cause P_{ineq} to differ as little as possible from mg and, moreover, to ensure that the natural oscillation frequency k of the mechanism will be sufficiently below

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(so that λ_1 can be close to unity).

Section 2. Possible Displacements of the Center Position of an Instrument Mechanism under Oscillation due to Vibrations, Resulting From a Variability of the Transmission Ratios From the Points of Origin of the Perturbation Forces in Vibrations to the Point to Which the Forces Are Reduced

The nonlinearity of the equation of motion of mechanisms, and in general the variability of the magnitude of $P_{ineq.}$ in eq.(3.2) may result in a displacement of

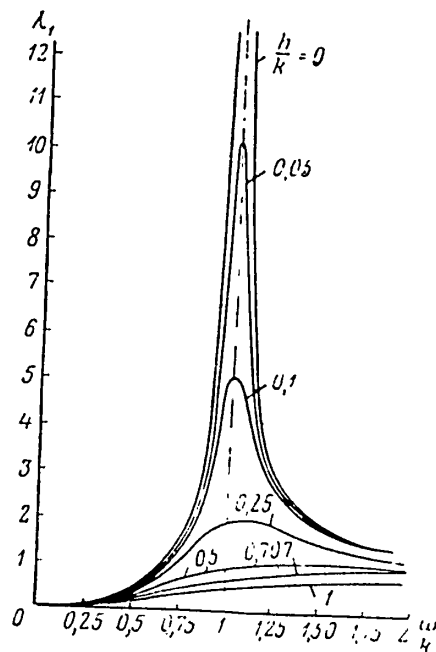


Fig.17 - The λ_1 Factor (in Harmonic Vibration) as a Function of the Ratio $\frac{\omega}{k}$ of the Frequencies of Forced to Natural ("Undamped") Vibrations, as a Function of the Damping $\frac{h}{k}$.

the midpoint of the mechanism when vibrations occur. Although the displacement at the center point, due to this cause is usually very small, we will discuss this question briefly. This is useful if only for the reason that it has been common, in dealing with the changes in mean instrument readings during vibrations, to refer to the so-called "Maxwell effect", which is one of the special cases of the effect of these factors, although in the majority of cases this explanation is not justified.

The changes in question, STAT occurring in eq.(3.2), may be due to any of the terms of that equation, but the greatest displacement,

ment of the center position of the mechanism (although this, too, is usually very slight) is primarily a result of a variability of the P_{ineq} factor in the perturbation term on the right side of the equation. We will, therefore, begin with this case.

As indicated above, the forces of inertia arising at various points in the mechanism during transferred accelerations are reduced to a single common point by multiplication by the transmission ratio of the rate of displacement in the direction of vibration of the point in which the force of inertia arises to the rate of displacement of the point to which all forces are shifted. However, in many mechanisms these transmission ratios are not constant and constitute functions of the relative position of the mechanism. In these cases, relative displacements of the mechanism during vibration cause the transmission ratios of separate parts of the mechanism to undergo constant changes in some periodic fashion. (True, the change is not great, since the relative displacements of the mechanism during vibration are usually small.) In this case, it may appear that the transmission ratio is on the average somewhat greater when the forces of inertia are directed one way than when they are directed in the other. As a result, the overall outcome may be some excess force in one direction; this causes a deviation of the center position of the mechanism, in the presence of vibration, from its position in the absence of vibration.

A special case of this phenomenon - deviation of the center position of a physical pendulum when the point of suspension is vibrating - was mentioned by Maxwell and has therefore come to be known as the "Maxwell effect". In this case, the perturbing force of inertia, when vibration is applied to the axis of the pendulum, creates a disturbing moment, M_i with respect to this axis, equal to

$$M_i = lP \sin \beta \frac{1}{g},$$

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where l is the distance between the center of gravity of the pendulum and its axis;
 P is the weight of the pendulum;

β is the angle between the direction of vibration and a perpendicular OA , dropped from the center of gravity of the pendulum A onto its axis O ;
 j is the component of transferred accelerations, normal to the axis of the pendulum.

In oscillations of the mechanism due to vibrations, the direction of the line OA changes, causing a change in the magnitude of the angle β . Thus, we have a variable transmission ratio $\frac{1}{l \sin \beta}$, between the rate of displacement of the center of gravity of the pendulum in the direction of the inertia force and the corresponding rate of rotation of the pendulum. This condition is known as the deviation of the center position of the pendulum from the vertical when vibration of its base is present.

This effect of vibration on a physical pendulum (the "Maxwell effect") is illustrated in Fig.18. Here O is the axis of the pendulum, and A is its center of gravity. Let us assume that we have a vibration of the point O (the suspension of the pendulum) in the direction O_1O_2 , in accordance with

$$f = a \sin \omega t,$$

where f is the deviation of the point of suspension from its center position;
 ω and a are constants.

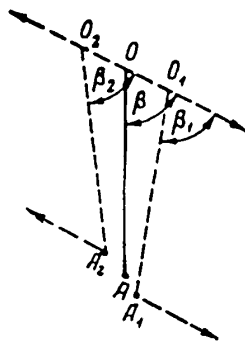


Fig.18 - Explanation of the Physical Principle of the Maxwell Effect

Let the direction OO_1 form an angle $\beta < \frac{\pi}{2}$ with the straight line OA . Let us consider $\omega \gg k$, where k is the natural frequency of the "undamped" oscillations of the pendulum. As a result, the phase shift between swings of the pendulum and the periodic vibrations of the perturbing force is equal approximately $\pi/2$. Therefore, when the point O is deflected to the right, the pendulum

occupies approximately the position O_1A_1 , while in deflections to the left it occupies approximately the position O_2A_2 . This being the case, in positions in which the point of suspension of the pendulum is to the right of its center point, the point O will have a leftward acceleration toward the center point, so that the pendulum acquires a counterclockwise torque. This moment is equal to $pl \sin \beta_1$. Similarly, for positions of the point O to the left of the center point, the pendulum will have a moment tending to rotate it clockwise and equal to $-pl \sin \beta_2$.

However, under conditions in which the suspension O of the pendulum, is to the right of its center point, the β angle will be somewhat greater than under conditions in which the suspension is to the left. Therefore the moments, periodically alternating in direction, will be of a somewhat greater magnitude in the counterclockwise than in the clockwise direction. To compensate for this difference, some additional moment should be applied to the pendulum.

All statements relative to the effect of the variability of the reduced force $P_{ineq.}$, in the presence of vibration, can be demonstrated analytically as follows: The magnitude of $P_{ineq.}$, entering as a multiplier into the right side of eq.(3.2), is equal to

$$P_{ineq.} = \sum \left[p_n \frac{dy_n}{dx} \right],$$

where p_n is the weight of some part n of the mechanism;

dy_n is the simple relative displacement of the center of gravity of this part in the direction of acceleration, corresponding to the simple displacement dx of the point in the mechanism to which all forces are reduced.

The transmission ratios $\frac{dy}{dx}$ between the velocities $\frac{dy}{dt}$ of displacement of the center of gravity of individual part in the direction of vibration and the corresponding velocity $\frac{dx}{dt}$ may vary from one position of the mechanism to the next. In other words, the displacement y is often some nonlinear function of the displacement x. Thus, the perturbation time of the right side of eq.(3.2), in the general

case, is a function of time (counting the acceleration j as some function of time), as would be the function of the desired magnitude x , which would yield a different solution of eq.(3.2).

A more detailed theoretical study of this case shows that in harmonic vibrations ($f = a \sin \omega t$, where f is the transferred displacement of the instrument body on vibration, while a is the vibration amplitude, and ω its frequency), the approximate magnitude of the mean force F_B of the excess action of the reduced inertia forces in one direction, are approximately equal to

$$F_B \approx - \frac{a^2 \omega^4 (\omega^2 - k^2) \sum \left[p \left(\frac{d^2 y}{dx^2} \right)_{x=x_0} \right] \sum \left[p \left(\frac{dy}{dx} \right)_{x=x_0} \right]}{2Mg^2 [(\omega^2 - k^2)^2 + 4\omega^2 h^2]}, \quad (3.4)$$

where M is the reduced mass of the entire mechanism;

$\frac{h}{k}$ is the degree of damping;

x_0 is the magnitude of x at the center point of the mechanism during vibrations.

This force F_B produces a displacement of the center point of the mechanism (corresponding to $x = x_0$ during vibration and differing from x_c by the corresponding position of the mechanism in the absence of vibration), while it is balanced by the force $F_{\text{mean}} - F_p$ where, as indicated above, F_{mean} is the magnitude of the position force at $x = x_0$, while $F_p = F$ is the magnitude of this position force at $x = x_c$, in the absence of vibration.

Thus, having determined the magnitude of F_B , it is easy to calculate the change $\Delta x = x_0 - x_c$ in the quantity x , and from this the change in the mean instrument readings.

However, if all the forces and moments are reduced to some specific axis of the mechanism, the magnitude of the mean moment M_B , displacing the center point of the mechanism, is approximately equivalent to

$$M_B \approx - \frac{a^2 \omega^4 (\omega^2 - k^2) \sum \left[p \left(\frac{d^2 y}{d\varphi^2} \right)_{\varphi=\varphi_0} \right] \sum \left[p \left(\frac{dy}{d\varphi} \right)_{\varphi=\varphi_0} \right]}{2Jg^2 [(\omega^2 - k^2)^2 + 4\omega^2 h^2]}, \quad (3.4a)$$

where φ is the angle of rotation of the axis to which the forces and moments are reduced;

J is the reduced moment of inertia of the entire mechanism.

In the majority of cases, these deflections of the center point of the mechanism, due to vibration, are quite small and are usually almost imperceptible. Likewise, the amplitude and nature of the oscillations of the mechanism during vibration can usually not be differentiated from the amplitudes and nature of oscillations at constant $P_{ineq.}$ in eq.(3.2). This is understandable, since the oscillation amplitude of the mechanism during vibration is usually small, so that the transmission ratios between the velocities of motion of various elements of the mechanism will not vary much, even if this transmission ratio is variable. Only at very great vibration intensity and high resultant accelerations will the above effect become noticeable.

Moreover, under certain circumstances and in specific instruments, this effect may yield a greater magnitude. Thus, in correction devices of certain gyroscopic instruments, very short pendulums have sometimes been used (in particular, pendulum dampers, in which the center of gravity is about 1 cm from the axis of suspension, are used in gyro horizons). Calculation shows that even a relatively slight vibration, with maximum accelerations of 2 or 3 G, at an angle of 45° to the vertical (this angle being very close to that at which the deviation of the center point of the pendulum, due to vibration, reaches its maximum) may deflect the center point STAT of such a pendulum by an angle of about 1° . Any deviation of the pickup of the correction device obviously results in the same deviation of the gyroscope.

If we take as our example a physical pendulum of 9.8 cm length and increase its vibration parameters somewhat, assuming 0.8 mm as the amplitude of vibration and

55 cycles as the frequency (the maximum acceleration then being about 10G), the deviation of the center point on vibration, at an angle of 45° to the vertical, may attain $1/48$ radian, or about $1^\circ 11.5'$, while the amplitude of the forced vibrations of the pendulum will be about $19'$. Thus, the pendulum will never, even in the extreme positions of its swing, reach a vertical position.

Further it must be noted that, in accordance with theoretical studies, the phenomenon under examination generally has a slightly more noticeable effect on pendulum-type instruments, while on instruments with more complex mechanisms the effect of the vibration in question is usually considerably less.

Here, in passing, we may point to another interesting result of the theoretical investigation on the behavior of a physical pendulum under vibration of its point of suspension. This finding has been fully confirmed by experiment. It was found that, when the vibration has a very high intensity (greatly exceeding the practically possible intensity of vibration of instruments in use), a pendulum has not two but four points of equilibrium: two stable and two unstable positions. Thus, if the vibration is vertical in direction, vibrations not exceeding some given intensity (depending on the parameters of the pendulum), will cause the pendulum to remain in the same position of equilibrium as in the absence of vibration (or, in other words, the "Maxwell effect" is lacking in this case). However, after reaching the given (very high) intensity of vibration in the pendulum, the stable downward vertical position of equilibrium will be supplemented by a stable upward vertical position of equilibrium, although this had hitherto been unstable. In addition, two positions of unstable equilibrium in the transverse direction appear*.

* The generation of this phenomenon requires a very high vibration intensity, hardly possible in practice. Thus, for a pendulum of 9.8 cm length, with a 1 mm vibration amplitude, this phenomenon initially occurs at a frequency of about 225.1 cycle STAT (which is equal to about 13,510 cycles per minute) corresponding to a maximum acceleration on vibration of about 200G, while at a vibration amplitude of 5 mm (cont'd)

If we have a vibration directed horizontally, of an intensity smaller than some given limiting intensity, the positions of intensity remain the same as in the absence of vibration (i.e., the "Maxwell effect" is also equal to zero). However, after this limiting intensity of vibration has been reached, the vertical downward position of equilibrium (hitherto stable) also proves unstable as does the vertical upward position. Now two new positions of stable equilibrium of the pendulum appear in the transverse direction*.

Section 3. Possible Displacement of the Center Point of an Instrument Mechanism in Oscillations During Vibrations Due to the Variability of the Reduced Mass of the Mechanism

In the simplest type of pendulum instrument (physical pendulums, Ceiger pendulums, magnetic compass, and various electric instruments with only a single moving part) in which, in writing the equations of motion, all powers and moments are reduced to the axis of the pendulum, we have a constant moment of inertia J in the term $J\ddot{\phi}$, constituting the moment of inertia in the relative motion of the pendulum as it appears in the equation of motion. The same can be said of more complex mechanisms when the transmission ratios between the corresponding displacements of all the parts of the mechanism are constant in magnitude. However, in many mechanisms consisting of a number of interacting parts, these transmission ratios may be variable, i.e., they may constitute functions of the displacement x of the point to which the forces are reduced. Due to this and to the reduced mass (or reduced moment of inertia) of the entire mechanism, the term of the equation representing

(cont'd) (clearly most improbable), it begins at a frequency of about 45.08 cycles (equal to about 2705 cycles per minute), corresponding to a maximum acceleration of about 40.1 G.

* This requires approximately the same vibration parameters as indicated for the case of vertical vibration.

the forces of inertia in relative motion of the mechanism, also may prove to be not a constant, but some function of the displacement x of the point to which the forces are reduced.

It is easy to understand that the latter circumstance may produce an effect of the same order as the deflection of the mechanism due to variability of transmission ratios caused by disturbing forces, which we described above.

The recovery factors of the position force of the mechanism may also be variable: i.e., the magnitude S in the term Sx of eq.(3.2) for example, which is the reduced rigidity of all elastic elements in the mechanism. However, as shown in more detailed investigations, this factor usually results only in a much less pronounced displacement of the mean position of the mechanism during vibration than the displacement produced by the above-described variables in the transmission ratio, due to the disturbing forces, and the variability in the reduced mass (or the reduced moment of inertia) in the mx term of eq.(3.2), which will be discussed below.

In addition to other considerations which offer a stronger confirmation of this conclusion, the following simple consideration can be made: Let the recovery factor S be variable and constitute some function of the displacement x . In this situation, it will fail to change its sign, since, in instruments, regardless of the positions of their mechanisms, the total recovery factor of the position forces is always positive; if the opposite were the case, there would be no stable equilibrium positions of the mechanism at various readings of the instrument. In order for the mean magnitude of the term Sx to equal zero when the mechanism is in vibration, i.e., in order for the force expressed by this term to act identically (on the average) in both directions, it is obviously necessary to take as the starting point of measurement a magnitude x , lying somewhere between the two extreme positions of the mechanism while it oscillates; in this manner, as the mechanism oscillates, its deviation x from the given starting point of the reading will be somewhat larger in one direction than in the opposite direction.

Thus, the displacement of the midpoint of the mechanism during vibration will in any case not exceed the bounds of the vibration amplitude of the mechanism; this displacement will represent a very small part of that amplitude, since in the oscillation range of the mechanism, it is impossible for the S factor to vary to any considerable degree.

It may be concluded on further thought, that the effect of the variability of the transmission ratio, produced by the damping element must also be substantially less than the effect produced by instability of transmission ratios, due to the perturbing forces and the effect due to the variability of the reduced mass.

In addition there are considerations which make it possible to conclude that a variability in the transmission ratio due to the damping force should, in general, not affect the midposition of the mechanism when vibration is present, if the damping force is proportional to the velocity to the first power.

If, at a point ϵ in the mechanism, a damping force represented by the expression is present $H\dot{\Delta y}$ is present, where H is the damping coefficient and Δy the displacement of the point ϵ , the damping force F_D , when reduced to some other point, will have the form

$$F_D = H \left(\frac{dy}{dx} \right)^2 \dot{\Delta x},$$

where Δx is the shift of the point D.

If we expand the quantity $\left(\frac{dy}{dx} \right)^2$ into a Taylor series on the variable Δx , we will have

$$F_D = H \left\{ \left(\frac{dy}{dx} \right)^2_{x=x_0} \dot{\Delta x} + \left[\frac{d \left(\frac{dy}{dx} \right)^2}{dx} \right]_{x=x_0} \Delta x \dot{\Delta x} + \frac{1}{2} \left[\frac{d^2 \left(\frac{dy}{dx} \right)^2}{dx^2} \right]_{x=x_0} \Delta x^2 \dot{\Delta x} + \dots \right\}. \quad (3.5)$$

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We see that all the terms of the series include, as a factor, a quantity of the form $\Delta x^n \dot{\Delta x}$. However,

$$\int_0^t \Delta x^n \dot{\Delta x} dt = \frac{1}{n+1} \Delta x^{n+1} + c$$

and since, in steady-state vibration, the magnitude of Δx fluctuates within given limits, then

$$\left(\frac{\int_0^T \Delta x^n \dot{\Delta x} dt}{T} \right)_{T \rightarrow \infty} \Rightarrow 0,$$

i.e., no force results capable of moving the midpoint of the mechanism.

In view of the foregoing, and allowing for the fact that even the effect of the variability in transmission ratios due to perturbing forces is itself quite insignificant, and that the effects of the variability in the transmission ratio due to the forces of inertia relative to motion are of approximately the same order, we will limit the discussion merely to indicating the approximate order of possible magnitude of this last effect.

We know that, in the presence of some relative acceleration x , at a point c of the mechanism for which a force equation is compiled, the total reduced force of inertia of the entire mechanism takes on the form

$$F_i = \sum \left[m_n \left(\frac{d\bar{y}_n}{dx} \right)^2 + J_n \left(\frac{d\psi_n}{dx} \right)^2 \right] \ddot{\Delta x} + \sum \left[m_n \left(\frac{d^2 \bar{y}}{dx^2} \right) \left(\frac{d\bar{y}}{dx} \right) + J_n \left(\frac{d^2 \psi_n}{dx^2} \right) \left(\frac{d\psi_n}{dx} \right) \right] (\dot{\Delta x})^2 \quad (3.6)_{\text{STAT}}$$

(the general form of an expression for the terms of the equation of motion of the mechanisms is derived in the Chapter on dynamic instrument errors).

Note: The term $\bar{y}$ in eq.(3.6) and the term y in the preceding Subsection are, naturally, not identical, since the quantity $\frac{d\bar{y}}{dt}$ in eq.(3.6) is the total relative linear motion of the center of gravity of the part, while in the above Subsection the quantity $\frac{dy}{dt}$ embraces only the velocity component which is operative in the direction of the transferred acceleration.

The second term in this expression should induce some shift of the center point of the mechanism in the presence of vibration, since its sign is always the same, regardless of the signs of the quantities Δx or $\dot{\Delta x}$. However, as shown by research, twice as great a shift, but in the opposite direction, is induced by any variation in the quantities $\left(\frac{d\bar{y}}{dx}\right)^2$ and $\left(\frac{dy}{dx}\right)^2$ in the first term, meaning that, in the final total, the approximate expression of the total reduced force of inertia of the entire mechanism developed during the relative motion takes on the form

$$F \approx \sum \left[m_n \left(\frac{d\bar{y}_n}{dx} \right)^2 + J_n \left(\frac{d\psi_n}{dx} \right)^2 \right]_{x=x_0} \bar{\Delta x} - \sum \left[m_n \left(\frac{d^2\bar{y}_n}{dx^2} \right) \left(\frac{d\bar{y}_n}{dx} \right) + J_n \left(\frac{d^2\psi_n}{dx^2} \right) \left(\frac{d\psi_n}{dx} \right) \right]_{x=x_0} (\dot{\Delta x})^2. \quad (3.7)$$

A more detailed study of this question by the author has shown that the effect of variability in the reduced mass is of approximately the same order as the effect of variation in the transmission ratios due to perturbing forces. These two effects, in most cases, do not influence each other (except for the fact that one cause, i.e., a variation in the transmission ratios $\frac{dy}{dx}$ is responsible for both phenomena under examination).

In addition, a study of a few examples substantiates the assumption that, frequently, the effect of variation in the reduced mass is somewhat smaller than the effect of variation in the transmission ratios due to perturbing forces. The opposite site may also occur.

In a number of special cases, the effect of variability in the transmission ratios caused by disturbing forces and the effect due to variations in the reduced

mass are of the opposite sign and therefore compensate each other to some degree (but only partially, of course, since they are usually not equal). Sometimes these effects are additive, rather than tending to counterbalance each other.

Section 4. Possibility of Periodic Disruption of the Given Kinematic Connection Between Parts of the Mechanism on Vibration, and Resultant Changes in Instrument Readings

Certain instruments sometimes show marked deviations from average readings when under vibration. However, in accordance with the foregoing, the effects due to the influence of nonlinearity in the equations of motion of instrument mechanisms are practically unnoticeable in the majority of such instruments. Therefore, shifts of the center point of the mechanism (and sometimes of a part of the mechanism) under vibration, in the majority of cases, can obviously be explained by other causes.

Such changes in instrument readings during vibration are also observed in a number of instruments with relatively simple mechanisms, which, when properly assembled, can be considered systems with one degree of freedom (i.e., climb indicators, tachometers, altimeters, and certain electric instruments). It is obvious that, in each of these instruments, some reason exists for the shift of the center point of the mechanism (or a portion of the mechanism) on vibration.

These reasons may vary from one design of an instrument to the next. In particular, in instruments whose mechanism includes several transmission connections in sequence, vibration may cause a periodic disruption of the regular, constant kinematic connection between parts of the mechanism, if the tension moment of the hair spring is inadequate. This is most likely to occur when the method of balancing the pickup element of the instrument is that shown in Fig.27.

If the hair spring in the instrument has been calculated inaccurately or if inadequate initial tension was provided during assembly (and, in general, if the conditions given in Chapter X are not met), then, at certain points in the mechanism, a periodic disturbance of the constancy of the kinematic connection will occur;

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as a result, specific portions of the mechanism will clash from time to time.

In view of the fact that, in the portion of the mechanism which includes the axis of rotation of the indicator, these deviations occur in one direction only, a shift in the center point of the indicator (usually accompanied by a rather large vibratory swing of the needle) may take place here.

To prevent this, the hair spring must be under adequate tension no matter what the position of the mechanism. In other words, the parameters of the hair spring (dimensions, material, initial tension) must be such as to satisfy the conditions of eq.(10.10), as will be proved in the Section on the analysis of instrument hair springs. If this condition is not satisfied, periodic disruptions of the kinematic connections in the mechanism may take place.

The present author has performed theoretical investigations of this phenomenon*, supported by experimentation, for the case of steady-state oscillations of a mechanism, i.e., of a steady-state condition with harmonic linear forward vibration of the instrument at a constant frequency ω and an amplitude a .

In such a steady state, if the conditions of eq.(10.10) are not satisfied, it may happen (and this is evidently the most frequent possibility) that, for each period of vibration, the two parts of the instrument mechanism between which connection has been broken will clash, each time in the same place. Of course, other cases may also occur, in which two parts of a mechanism clash several times in the course of a single period of vibration, or a case in which the full cycle of the phenomenon under examination has a duration equal to several periods of vibration, i.e., the entire sequence of the phenomenon is repeated only when an interval of time has elapsed that may be expressed by $n \frac{2\pi}{\omega}$, where n is not unity but some other integer (2,3 etc.). However, in the studies referred to, only that case was examined - for the sake of simplicity - in which a full cycle of the phenomenon was equal to one period of vibration, during which the two parts of the mechanism clashed only once.

* Similar studies have been conducted by V.A.Shcherbakova, Engineer.

As was to be expected in view of the nature of this phenomenon, the differential equations resulting from theoretical examination of this question have to be solved by the usual methods of alignment. We will not derive these equations and their solutions here since, in designing and calculating instruments, it is sufficient to adhere only to the conditions of eq.(10.10) to prevent this phenomenon from occurring at all.

Theoretical studies have established that the solutions of equations, based on the assumption that the two parts of the mechanism clash only once during a single period of vibration, do not always correspond to actuality. Such correspondence depends on the parameters of the mechanism and the parameters of vibration. Several conditions covering the possibility or impossibility of occurrence of the given case were worked out in the investigations in question.

It should also be noted that numerical calculations on the basis of these theoretical studies show that, with respect to real values of solutions obtained in this manner, comparatively significant variations in instrument readings may occur, of primarily the same order as those found in practice when the indications are changed by vibration.

However, if the solutions obtained by means of equations derived in these investigations do not correspond to reality, we are obviously dealing with the cases indicated above, in which the two parts of the mechanism in question clash several times during a single period of vibration, or in which the total cycle of the phenomenon is of a duration equal to several periods of vibration. These cases may be resolved by similar methods of alignment; however, here the number of unknowns increases and, correspondingly, also the number of equations increases (obviously by the same factor as the number of clashes of the parts of the mechanism during a single complete cycle). In other words, a more complex system of equations for solution of the system results. In the research in question, no such cases were studied; however, an analysis should reveal the changes in instrument readings

caused by vibration to be of approximately the same order as in the case examined above.

Moreover, even in instruments in which the hair spring tension is inadequate, there may also be other causes giving rise to a shift in the center point of the indicator when vibration occurs. In general, the source of the phenomenon under discussion is usually some particular shortcoming in design.

Section 5. Frictional Entrainment in Cases of Elliptical Vibration

A source giving rise to displacement of the center point of the mechanism during vibration, and common to all instruments, is the entrainment by friction in circular or elliptical vibration. Let us take as an example a part rotating about a horizontal axis. Let us assume, in this case, the existence of vertical circular vibration in the vertical plane normal to the axis of the part.

Let us examine this phenomenon for a case in which $\rho\omega^2 > g$. Let the center O_1 of the bearing ABC in Fig.19 describe, during vibration, a circle O_1DE of a radius ρ about the point O. Point O_2 is the center of a pin that, at the given instant, is forced against the bearing at the point A (the angle ρ by which the straight line O_1O_2 lags in the direction OO_1 results from the force of friction $N\mu$). The line OH is the vertical direction.

Then the force N of compression at the point of contact A is equal to

$$N = p \cos(\gamma - \beta) + \rho \frac{p}{g} \omega^2 \cos \beta + (r_n - r_0) \frac{p}{g} (\omega - \dot{\beta})^2. \quad (3.8)$$

The force equation in the perpendicular direction will assume the form

$$\rho \frac{p}{g} \omega^2 \sin \beta - p \sin(\gamma - \beta) - N\mu = 0, \quad \text{STAT} \quad (3.9)$$

where p is the weight of the part (considering its center of gravity to be on the O_2 axis);

r_n is the radius of the bearing;

r_o is the radius of the pin (the difference $r_n - r_o$ usually being quite small;

μ is the coefficient of friction.

From eqs.(3.8) and (3.9) we derive

$$\begin{aligned} & \left| N - (r_n - r_o) \frac{p}{g} (\omega - \dot{\beta})^2 \right|^2 + (N\mu)^2 = p^2 + \left(\rho \frac{p}{g} \omega^2 \right)^2 + \\ & + 2\rho p \frac{p}{g} \omega^2 [\cos(\gamma - \beta) \cos \beta - \sin(\gamma - \beta) \sin \beta] = \\ & = p^2 + \left(\rho \frac{p}{g} \omega^2 \right)^2 + 2\rho p \frac{p}{g} \omega^2 \cos \gamma \end{aligned}$$

or

$$\begin{aligned} & N^2 (1 + \mu^2) - 2(r_n - r_o) \frac{p}{g} (\omega - \dot{\beta})^2 N - \\ & - \left\{ p^2 + \left(\rho \frac{p}{g} \omega^2 \right)^2 + 2\rho p \frac{p}{g} \omega^2 \cos \gamma - \right. \\ & \left. - \left| (r_n - r_o) \frac{p}{g} (\omega - \dot{\beta})^2 \right|^2 \right\} = 0, \end{aligned}$$

from which

$$N = \frac{(r_n - r_o) \frac{p}{g} (\omega - \dot{\beta})^2 + \sqrt{p^2 + \left(\rho \frac{p}{g} \omega^2 \right)^2 + 2\rho p \frac{p}{g} \omega^2 \cos \gamma}}{1 + \mu^2} \quad (3.10)$$

(we do not make use of the solution with the minus sign in front of the radical since the magnitude of the radical will in this case probably be larger than that of the first term of the numerator).

Thus, a moment of friction between pin and bearing M , always acting in the same direction, is applied to the axis of the pin. This moment is equal to

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$$M = r_0 \mu N.$$

Substituting ωt for the angle γ in eq.(3.10), we obtain a mean quantity M_{cp} for this moment, equal to

$$\begin{aligned} M_{cp} &= \frac{r_0 \mu \int_0^{\frac{2\pi}{\omega}} N dt}{\frac{2\pi}{\omega}} = \frac{r_0 \mu}{1 + \mu^2} \frac{\omega}{2\pi} \left[\frac{2\pi}{\omega} (r_n - r_0) \frac{p}{g} (\omega - \beta)^2 + \right. \\ &\quad \left. + \int_0^{\frac{2\pi}{\omega}} \sqrt{p^2 + \left(\rho \frac{p}{g} \omega^2 \right)^2 + 2p\rho \frac{p}{g} \omega^2 \cos \omega t} dt \right] = \\ &= \frac{r_0 \mu}{1 + \mu^2} \frac{\omega}{2\pi} \left[\frac{2\pi}{\omega} (r_n - r_0) \frac{p}{g} (\omega - \beta)^2 + \right. \\ &\quad \left. + \int_0^{\frac{2\pi}{\omega}} \sqrt{\left(p - \rho \frac{p}{g} \omega^2 \right)^2 - 4p\rho \frac{p}{g} \omega^2 \sin^2 \frac{\omega t}{2}} dt \right]. \end{aligned} \quad (3.11)$$

The second term of the expression under the radical sign and within the brackets is an ordinary elliptical integral within limits from 0 to π , so that its magnitude can be determined by means of available Tables, from which the magnitude of M_{cp} can be calculated.

However, even without this calculation we see that the mean value (for the duration of a complete revolution of the point C_1 about the point O or, in other words, for a time of $\frac{2\pi}{\omega}$ of the numerator on the right side of eq.(3.10), even if we ignore the first term of this numerator (which is usually quite small), is somewhat larger than the quantity $\rho \frac{p}{g} \omega^2$. This is the reason for the fact that the denominator contains the term $1 + \mu^2$. STAT

From the above statements the following conclusion can be drawn. The average moment of entrainment M_{cp} developed about the axis O_2 during circular unilateral vi-

brations, is approximately equal to, and often even slightly larger than, the moment of friction due to the weight of the part (in the absence of vibration) multiplied

by the ratio $\frac{\rho \omega^2}{g}$.

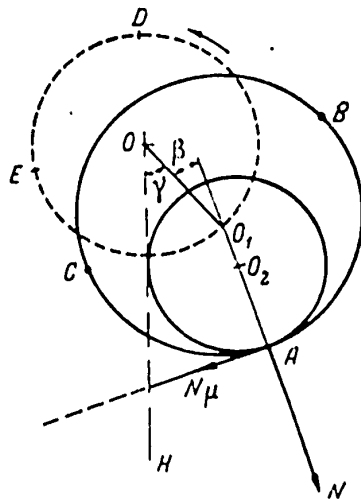


Fig.19 - Entrainment by Friction $N\mu$
on Circular Vibration

A similar phenomenon which will also occur is $\rho \omega^2 < g$, i.e., when the axis of the part, on vibration, is not separated from the lower surface of the bearing. Despite the fact that, under these conditions, only a minor slippage to one side or the other relative to the surface of the bearing occurs, the force with which the axis is pressed to the surface of the bearing will be stronger when the axis slips in one direction than when it slides in the other; this is due to the fact that, on motion in one direction, the centrifugal

force is added to the force of gravity, while in the other direction the former is subtracted from the latter. Therefore, the moment of deviation in one direction will be greater than in the opposite.

We will not here derive the equation for calculating the magnitude of M_{cp} for this particular case, since this derivation, being more complex than in the above case, is nevertheless less important in terms of the phenomenon under examination. The point is that the smaller the magnitude of $\rho \omega^2$, the smaller will be the moment M_{cp} of entrainment by friction during circular vibration. However, even when the magnitude is $\rho \omega^2 = g$, the entrainment by friction is usually rather slight.

As investigations have shown, when the axis is vertical and the circular STAT tion is horizontal we find approximately the same magnitude for the moment of entrainment by friction during circular vibration, i.e., this moment is, in this case

as well, approximately equal to the moment of perpendicular friction of the part (in the absence of vibration), multiplied by the ratio $\frac{e\omega^2}{g}$. Here again, it may be slightly larger than that magnitude.

Thus, entrainment by friction during circular vibration may sometimes be considerable, since the frictional error in the absence of vibration may, with some instruments, be of noticeable, although small, magnitude. In circular vibration, the moment of entrainment is as many times larger as the moment of friction of the given part in the absence of vibration, since the centrifugal acceleration $\rho\omega^2$ during vibration is greater than the acceleration of gravity g .

It should be noted that tests on the effect of entrainment by friction during elliptical vibration, conducted with standard diaphragm aircraft instruments, revealed no noticeable variations in the instrument readings. This is explained primarily by the fact that the parallel axes of meshing parts in such instruments are linked in such fashion that one and the same circular vibration has the effect of creating moments in two adjacent axes that tend to displace the mechanism in opposite directions. Therefore, these drift moments tend to compensate each other to a considerable extent. Secondly, the experiments were not so conducted as to exclude the possibility of vibrations in directions parallel to these axes, which would of course greatly reduce the magnitude of the force of friction.

This error may reach particularly high values in the magnetic system of instruments of the old G1 design, in which the force of recovery is quite small as compared to the relatively high weight of the system. If, for example, the weight of the magnetic system is of the order of 5 gm, the axial radius about 0.2 mm, the coefficient of friction $\mu = 0.15$, and $\rho\omega^2$ about 2.5 G, the moment of deviation may reach a magnitude of about 0.375 gm-mm. However, even if proper shock absorbers for the instrument have been designed to prevent $\rho\omega^2$ from exceeding 1 G, the force of deviation may still reach 0.15 gm-mm. Nevertheless, the entire directive moment, when the system has deviated by 90° , is equal to approximately $\frac{0.18 \times 300}{981} =$

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= 0.055 gm-cm = 0.55 gm-mm (considering the magnetic moment of the rhumb card to be equal to approximately 300 in the CGS system).

Because of this, the axis of the magnetic system in the old GMT design was arranged as follows: One end formed the pin of a bearing fixed to the housing of the magnetic system, and the other carried a bearing into which a pin, fixed to the housing of the magnetic system, was inserted. As a result, the moments of deviation at the two ends of the axis were opposite in direction and compensated each other to a considerable degree.

The point is that the moment of entrainment produced by friction during circular vibration, which we are examining, has a direction opposite to that of the vibration in the case of an ordinary bearing, in which the end of the axis of a part forms a pin while the bearing is mounted in the housing of the system. In a case in which the bearing is at the end of the axis of the part, while the pin is fastened to the housing of the system, a moment of the same order will result, but one that tends to rotate the part in the direction of vibration.

For an ordinary magnetic compass, where a card immersed in a liquid is supported at only one point (its pivot) in the surface of its socket, it was found that deviations due to friction in the support, during circular vibration, are imperceptibly slight, despite the small force of recovery in such a compass. This is due to the insignificant magnitude of the radius of the area of contact between pivot and socket.

Investigations also revealed that no significant deviations can occur in the magnetic system, due to resistance of the liquid to conical unidirectional motion of the card in the bowl case, under this type of vibration.

However, individual cases of deviation of the needle have been observed in compass vibrations. Such deviation actually was somewhat larger than that resulting from one particular cause (to be discussed below) identified by H. Klotter. The following explanation of such deviations of compass needles may be advanced at this

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point: It would appear that, as the card moves relative to the compass bowl case, it tends to agitate the liquid which, as a result, begins to move relative to the bowl case. The movement is, however, restrained by the walls of the case and by other parts of the compass. The resultant currents developing in the liquid, in turn, may set up some moment in the card (due to viscous friction) relative to its vertical axis.

It would seem that this moment cannot be great, but the moment of recovery in the compass card is also very small. As a result it is quite possible that this factor may induce noticeable deviations in the compass card.

Section 6. Effect of Vibrations on Certain Mechanisms with Several Degrees of Freedom

One of the causes for shifts of the center point of the indicator, during vibration of ordinary magnetic compasses in which the magnetic system is supported at a single point (the end of the pivot) formed by the socket, was pointed out by F. Plotter in 1939.

In the given instance, the system has not one but three degrees of freedom, the center of gravity of the card immersed in the liquid being below the point of support and at a considerably greater distance l , than the displacement d of this center of gravity in a southerly direction. Therefore, during vibration, the oscillations of the card occur simultaneously about all three axes. However, the essence of the matter does not even lie in the fact that the Maxwell effects, resulting from the oscillations in a single direction, are capable of increasing this effect somewhat relative to another axis of rotation of the system. It is true that this may occur but, as obvious from a consideration of the foregoing, no significant deviations of the center point of the needle occur when the vibrations are of the order encountered in actual practice. However, here a phenomenon observed by F. Plotter occurs in addition. This is somewhat similar to the Maxwell effect but of a different origin.

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It should also be mentioned that the resultant deviation of the card, relative to its vertical axis, takes place regardless of whether the center of gravity of the card has been shifted from the vertical axis passing through the point of support or not. Thus, the deviation will occur even when there is no Maxwell effect relative to the vertical axis, and when virtually no oscillations about this vertical axis take place. The phenomenon under examination occurs only because the center of gravity of the card is below its point of support. In addition, as the center of gravity of the card is shifted below its point of support to a considerably greater distance than its displacement to the south side of the card, the magnitude of the deviation of the compass needle from the vertical axis is almost entirely independent of the magnitude of displacement of the center of gravity of the card toward its southern end.

By way of example, let us examine a simple linear horizontal sinusoidal vibration $f = a \sin \alpha t$, at an angle β to the mean compass bearing when vibration is present (here the angle is $\beta = \beta_0 - \delta$, where β_0 is the angle between the direction of vibration and the N-S line, and δ is the angle of deviation of the center point of the needle from this direction in the presence of vibration).

The oscillations of the card can be resolved down into oscillations about three mutually perpendicular axes passing through the point of support and coinciding, when the card is in its mean position during the vibration, with its three main axes of inertia drawn through the point O of the support, to wit:

- a) Oscillations about the vertical axis O;
- b) Oscillations about the horizontal direction (Fig.20), coinciding with the N-S line of the card and assuming a mean position of this line when vibration is in progress (axis Ox);
- c) Oscillation about a direction coinciding with the W-E line of the card (STAT assuming a mean position of this line during vibration (the Oy axis).

On the Oz axis, we assume the upward direction to be positive, while on the

Ox axis it is the southerly direction that is positive (S) and on the Oy axis, the easterly (E).

If, for purposes of simplification, we examine solely the oscillations of the card about the horizontal Ox axis or the horizontal Oy axis exclusively due to the effect of the corresponding component of the perturbation force perpendicular to the given axis, and if we make this examination independently of the oscillations about the other axes, i.e., independently of the mutual effects of the oscillations about

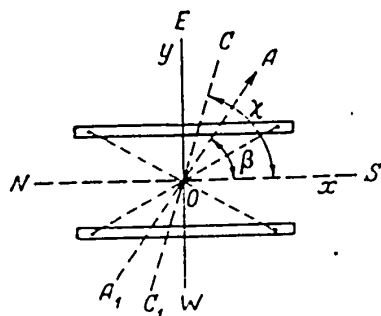


Fig.20 - Discrepancy of the Direction C_1C of Oscillation of the Center of Gravity of a Magnetic Compass Card, and the Direction A_1A of Horizontal Vibration

the Ox and Oy axes and of the influence of the oscillations about the Oz axis (a simplification which would appear to have a very minor effect on the final result), we obtain the following specimen rule for oscillations about the Ox and Oy axes.

If the magnitude f of periodic displacement in either direction of the point O of the support is considered positive in the direction shown by the arrow in Fig.20, the acceleration in the OE direction will be $-a\omega^2 \sin \beta \sin \omega t$, and in the OS direction it will be $-a\omega^2 \cos \beta \sin \omega t$. The forces of inertia resulting

from these displacements of the card (together with the point of support) in the directions indicated, are equal, respectively, to $\frac{p}{g} a\omega^2 \sin \beta \sin \omega t$, and $\frac{p}{g} a\omega^2 \cos \beta \sin \omega t$, where p is the weight of the card in the liquid.

Due to the periodic force $\frac{p}{g} a\omega^2 \sin \beta \sin \omega t$, oscillations develop about the axis OS, having approximately the following form (ignoring the Maxwell effect on vibration about the Ox axis):

$$\varphi_s \approx \frac{apl}{gJ_x} \lambda_1 \sin \beta \sin (\omega t - \delta_1). \quad (3.12)$$

Here

$$\lambda_1 = \frac{\omega^2}{\sqrt{(k_1^2 - \omega^2)^2 + 4\omega^2 h_1^2}}$$

and

$$\delta_1 = \arctg \frac{2h_1\omega}{k_1^2 - \omega^2},$$

where

$$k_1^2 = \frac{lp}{J_x} \text{ and } h_1 = \frac{B}{2J_x},$$

and B is the damping coefficient of the card as it oscillates about the horizontal axis.

Due to the periodic force $\frac{p}{g} a\omega^2 \cos \beta \sin \omega t$, oscillations develop about the OW axis (when viewed from the W end), having approximately the following form:

$$\varphi_W \approx \frac{apl}{gJ_y} \lambda_2 \cos \beta \sin (\omega t - \delta_2). \quad (3.13)$$

Here

$$\lambda_2 = \frac{\omega^2}{\sqrt{(k_2^2 - \omega^2)^2 + 4\omega^2 h_2^2}}$$

and

$$\delta_2 = \arctg \frac{2\omega h_2}{k_2^2 - \omega^2},$$

where

$$k_2^2 \approx \frac{lp}{J_y} \text{ and } h_2 = \frac{B}{2J_y}.$$

From eqs.(3.12) and (3.13) we see the following: If $J_x \neq J_y$ (which is usually the case in compass cards), then

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$$\frac{\tilde{z}_s}{\varphi_w} = \operatorname{tg} \chi \neq \operatorname{tg} \beta,$$

i.e., the direction of the oscillations of the center of gravity of the card does not coincide with the center of the vibrations. As a result, the forces of inertia during vibration create an additional moment relative to the vertical axis, while the arms $l\varphi_g$ and $l\varphi_v$ of the forces creating this additional moment change their direction almost simultaneously with the change in the signs of the forces of inertia $\frac{p}{g} a\omega^2 \sin \beta \sin \omega t$ and $\frac{p}{g} a\omega^2 \cos \beta \sin \omega t$, with the result that the overall total of the supplementary moments under examination always operates in the same direction.

Therefore, in the final analysis, there is some resultant moment causing the card to deviate to one side, as does the Maxwell effect, but this moment is already somewhat larger than the moment produced by the Maxwell effect and may induce a slightly greater deflection of the card.

It is possible to calculate the approximate magnitude of this moment.

Thus, in addition to the moment about axis Oz , periodically changing its direction from side to side and produced by the displacement of the center of gravity of the card toward the south, and in addition to the Maxwell effect discussed above, we have the following moments: The force of inertia $\frac{p}{g} a\omega^2 \cos \beta \sin \omega t$ creates a further moment, M_1 , equal approximately to

$$\begin{aligned} M_1 &\approx -l \frac{ap!}{gJ_x} \lambda_1 \sin \beta \sin (\omega t - \delta_1) \frac{p}{g} a\omega^2 \cos \beta \sin \omega t = \\ &= \frac{a^2 l^2 p^2 \omega^2}{g^2 J_x} \lambda_1 \sin \beta \cos \beta [\cos \delta_1 \sin^2 \omega t - \sin \delta_1 \sin \omega t \cos \omega t], \end{aligned}$$

and the force of inertia $\frac{p}{g} a\omega^2 \sin \beta \sin \omega t$ creates a moment M_2 , approximately equal to

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$$M_2 \approx l \frac{ap l}{g J_y} \lambda_2 \cos \beta \sin (\omega t - \delta_2) \frac{p}{g} a \omega^2 \sin \beta \sin \omega t = \\ = \frac{a^2 l^2 p^2 \omega}{g^2 J_y} \lambda_2 \sin \beta \cos \beta [\cos \delta_2 \sin^2 \omega t - \sin \delta_2 \sin \omega t \cos \omega t].$$

As a result we obtain a moment M , equal to

$$M = M_1 + M_2 \approx -\frac{a^2 l^2 p^2 \omega^2}{g^2} \sin \beta \cos \beta \left\{ \left[\frac{\lambda_1 \cos \delta_1}{J_x} - \frac{\lambda_2 \cos \delta_2}{J_y} \right] \times \right. \\ \times \sin^2 \omega t + \left[\frac{\lambda_1 \sin \delta_1}{J_x} - \frac{\lambda_2 \sin \delta_2}{J_y} \right] \sin \omega t \cos \omega t \Big\} = \\ = -\frac{1}{2} \frac{a^2 l^2 p^2 \omega^2}{g^2} \sin \beta \cos \beta \left[\frac{\lambda_1 \cos \delta_1}{J_x} - \frac{\lambda_2 \cos \delta_2}{J_y} \right] + \\ + \frac{1}{2} \frac{a^2 l^2 p^2 \omega^2}{g^2} \sin \beta \cos \beta \left[\frac{\lambda_1 \cos \delta_1}{J_x} - \frac{\lambda_2 \cos \delta_2}{J_y} \right] \cos 2\omega t - \\ - \frac{1}{2} \frac{a^2 l^2 p^2 \omega^2}{g^2} \sin \beta \cos \beta \left[\frac{\lambda_1 \sin \delta_1}{J_x} - \frac{\lambda_2 \sin \delta_2}{J_y} \right] \sin 2\omega t. \quad (3.14)$$

The second and third terms in expression (3.14), containing the factors $\cos 2\omega t$ and $\sin 2\omega t$, induce further oscillatory disturbance of the card about the Oz axis. Their action is in some manner added to the effect of the perturbation moment $H_V M_C \frac{a\omega^2}{g} \sin \beta \sin \omega t$ (where H_V is the vertical component of the magnetic field of the earth and M_C is the magnetic moment of the card) with the result that there may also be some change in the magnitude of the Maxwell effect, resulting (as indicated above) from the fact that, as the card oscillates about the Oz axis, there is an identical variation in the magnitude of the angle between the $N-S$ axis of the card and the direction of vibration. However, this change in the magnitude of the Maxwell effect is insignificant since, in our compasses, the factors at work with $\cos 2\omega t$ and $\sin 2\omega t$ in eq.(3.14) are considerably less than the quantity $H_V M_C \frac{a\omega^2}{g} \sin \beta$. However, the Maxwell effect created during oscillation of the card about the Oz axis, under the effect of the perturbation moment $H_V M_C \frac{a\omega^2}{g} \sin \beta \sin \omega t$, is itself so small as to be imperceptible.

However, the first term in eq.(3.14) is a constant moment, always acting in the same direction.

Since, in compasses, the quantities k_1 and k_2 are only a fraction of ω , the quantities $\cos \delta_1$ and of $\cos \delta_2$ are quite close to minus unity, while the quantities λ_1 and λ_2 are each almost exactly equal to plus unity. Therefore, the magnitude M is approximately equal to

$$M \approx \frac{1}{2} \frac{a^2 l^2 p^2 \omega^2}{g^2} \sin \beta \cos \beta \frac{J_y - J_x}{J_y J_x}. \quad (3.15)$$

If the compass card is housed in a liquid (as is the case in the designs of many compasses), there usually is a float (a hollow reservoir) at the top of the card. Therefore, the center of volume of the card is above its center of mass, imparting stability to the card when it is in a liquid, since the center of the mass of the card in these designs is often higher than the point of support of the card, while the center of volume is even substantially higher than the center of mass. In this case, the quantity lp entering into eq.(3.15) is equal to

$$lp = l_1 p_1 - l_0 p_0,$$

where p_0 is the weight of the card in vacuum;

l_0 is the distance from its center of mass to the point of support (counting as positive in the upward direction);

p_1 is the weight of a liquid of the same volume as the card;

l_1 is the distance from the center of volume of the card to the point of support.

The moment M_1 resulting from this phenomenon is added to the moment due to the Maxwell effect, but it is usually significantly greater than the latter. Let us take, for example, a card with two magnets. Here, as we know, the distance between the magnets is such that straight lines connecting the equivalent poles of the two magnets and drawn to intersect diagonally through the center of the card, will cut

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each other at an angle of 60° . Therefore, the ratio of the moments of inertia of the magnets themselves, relative to the axes Ox and Oy , is approximately $\frac{17}{24}$ to $\frac{18}{24}$. Considering that the cards consist not only of magnets but of other components whose moments of inertia, relative to the Ox and Oy axes, are approximately equal, it may be presumed that in many cards of this type, we have on the overall

$$\frac{J_y}{J_x} \approx \frac{12}{10}.$$

Assuming, by way of an example (in accordance with the parameters of aircraft compasses) that $J_y \approx 0.012 \text{ gm-cm-sec}^{-2}$; $J_x \approx 0.01 \text{ gm-cm-sec}^{-2}$; $p_0 = 8.1 \text{ gm}$; $p_1 \approx 7.1 \text{ gm}$ (i.e., the weight of the card in the liquid is approximately 1 gm); $l_0 \approx 1.6 \text{ mm}$; $l_1 \approx 4.3 \text{ mm}$; $\omega = 240 \text{ rad/sec}$; $a = 0.5 \text{ mm}$ and $\beta = 45^\circ$, we will have, approximately

$$\begin{aligned} M &\approx \frac{(7.1 \cdot 4.3 - 8.1 \cdot 1.6)^2 \cdot 10^4 \cdot 2.4^2}{2.4 \cdot 10^4 \cdot 981^2 \cdot 2} \cdot \frac{2 \cdot 10^6}{120 \cdot 10^3} \text{ g cm} = \\ &= \frac{17.57^2 \cdot 24}{4.981} \text{ dynes cm} \approx 1.89 \text{ dynes cm}. \end{aligned}$$

Assuming, by way of an example, that the magnitude of $H_2 k_c$ is approximately 54 dynes cm (where H_2 is the horizontal component of the earth's magnetic field), we obtain

$$\frac{M}{H_2 M_k} \approx \frac{1.89}{54} \approx 0.035 \text{ rad.}$$

Thus, with the approximate data used, the moment M may induce a deviation in the compass needle ϵ , of approximately

$$\epsilon = 0.035 \text{ rad} \approx 2^\circ.$$

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In his paper, K.Klotter derives considerably higher magnitudes for deviations of the compass needle. This probably can be attributed to certain inaccuracies in

his calculations, perhaps in the computation of the magnitude of ρ for a card immersed in a liquid.

Thus, the phenomenon described by K.Klotter may induce significant deviations of the compass needle. It must be noted, however, that in actual practice, even greater deviations of the compass needle have been induced by vibration than those due to the phenomena we have discussed.

It can be shown that the card of the magnetic compass is only a special example of a class of mechanisms (or systems) having several degrees of freedom, on which vibration exercises a similar effect. For example, imagine a system with two degrees of freedom and let x and y be coordinates corresponding to the two degrees of freedom of the system. In this case, a system is readily conceivable in which vibration will cause the equation for the x coordinates to have the following form:

$$m_1 \ddot{x} + H \dot{x} + Sx = N \sin \omega t,$$

where ω is the frequency of vibration;

N is a magnitude constituting a function of another coordinate in the system.

For example, let $N = cy$, where c is a constant. In this case it is entirely possible that vibration will also cause forced fluctuations in the magnitude of y , meaning that y may appear as follows:

$$y = A \sin \omega t + B \cos \omega t,$$

from which we obtain

$$m_1 \ddot{x} + H \dot{x} + Sx = c \cdot A \sin^2 \omega t + cB \sin \omega t \cos \omega t. \quad (2.16)$$

The term $c \cdot A \sin^2 \omega t$ in this equation induces a shift in the coordinates of x from zero, in a specific direction.

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Examples of such systems may readily be cited. Let us consider, for example, an ordinary physical pendulum (not a gyroscopic pendulum) hung from gimbals and so

inclined (as in a ship, tank, or aircraft) that the outer axis of the gimbals has ceased to be horizontal and forms an angle γ with the horizontal plane (Fig.21).

Let us assume that, in this situation, we have a horizontal linear vibration, whose direction NN_1 forms an angle ϵ with the projection of the outer axis of the gimbals in the horizontal plane.

As long as the angle $\gamma = 0$, the forced swings of the pendulum take place in the vertical plane, and the mean position of the pendulum along its swings is vertical.

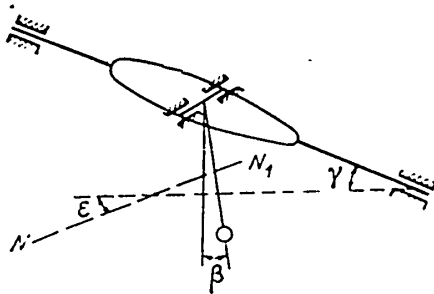


Fig.21 - Effects of Vibration on the Oscillation of a Pendulum on Gimbals, if the Outer Axis of the Latter is Not Horizontal

However, the direction of vibration may not be in the plane of oscillation of the pendulum, if the moments of inertia of the pendulum, relative to the two axes of the gimbals, are not identical. If the axis of the outer frame of the suspension forms an angle γ to the horizontal, vibration will cause formation of some angle of deviation of the mean position of the pendulum, relative to this axis of the outer frame; in other words, the mean position of the pendulum, during vibration, will no

longer be vertical.

Let us demonstrate this in approximate form: The equation for the oscillations of a pendulum about the axis of the inner frame of a suspension, if we ignore the effect (which is clearly very inconsequential) of the presence of the angles α of deviation of the pendulum relative to the axis of the outer frame of the suspension, may be written approximately as follows:

$$J_2 \ddot{\beta} + B_2 \dot{\beta} + l p \beta = \frac{l p}{g} \cos \epsilon a \omega^2 \sin \omega t,$$

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relative to the axis of its in-

is the distance of its center of gravity from the axis of the suspension;
 a is the vibration amplitude.

Therefore, the expression for forced oscillations about the axis of the inner frame appears as follows:

$$\beta \approx \frac{apl}{gJ_2} \lambda_2 \cos \varepsilon \sin(\omega t - \delta_2),$$

where, in view of the fact that ω is usually many times larger than the frequency of the natural oscillations of the pendulum, the dynamic susceptibility factor λ_2 is, in this case, very close to unity, and the phase shift δ_2 is very close to π .

The equation for oscillations about the axis of the outer frame will have approximately the following form:

$$J_1 \ddot{\alpha} + B_1 \dot{\alpha} + l \cos(\gamma + \beta) p \cos \gamma \alpha = - \frac{l \cos(\gamma + \beta) p}{g} \sin \varepsilon \omega^2 \sin \omega t. \quad (3.17)$$

Here, in view of the smallness of the angle β , we have

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$$\begin{aligned} \cos(\gamma + \beta) &= \cos \gamma \cos \beta - \sin \gamma \sin \beta \approx \cos \gamma - \beta \sin \gamma = \\ &= \cos \gamma - \sin \gamma \frac{apl}{gJ_2} \lambda_2 \cos \varepsilon \sin(\omega t - \delta_2). \end{aligned}$$

After substituting this expression in eq.(3.17), the right side of the equation will have, in addition to a perturbation term in the usual harmonic form, a term of

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the form

$$-\sin \gamma \sin \epsilon \cos \epsilon \left(\frac{a p l \omega}{g} \right)^2 \frac{\lambda_2}{J_2} \sin \omega t \sin (\omega t - \delta_2),$$

which provides a mean moment (averaged in point of time) about the axis of the outer frame of the suspension in one specific direction, since the magnitude δ_2 is close to π .

It is true that, in this case, the left side of eq.(3.17) will also contain a term

$$-\sin \gamma \cos \gamma \cos \epsilon (p l)^2 \frac{a \lambda_2}{J_2 g} \sin (\omega t - \delta_2) \alpha,$$

similar to the term causing the Maxwell effect examined above. However, there is no basis here for assuming that the displacements caused by these two effects [i.e., the effect due to the influence of the term containing the factor $\sin \omega t \sin (\omega t - \delta_2)$, and the effect of the term containing the factor $\sin (\omega t - \delta_2) \alpha$] will necessarily compensate each other, the more so as each of these two terms also contain independent factors; one of these contains $\sin \epsilon$ as a factor while the other contains $\cos \gamma$.

These last considerations have been made here to show that the phenomenon noted by K.Klotter exists not only in a pendulum with three degrees of freedom, but throughout a class of systems having several degrees of freedom.

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CHAPTER IV

PARTICULAR FACTORS TENDING TO REDUCE THE EFFECT OF VIBRATIONS ON
INSTRUMENT MECHANISMSSection 1. Selection of the Natural Oscillation Frequency of Instrument Mechanisms
and their Damping

Shock absorption is used in all instruments and is one of the most effective methods of counteracting the effects of vibration. Vibration absorbers are provided both for the instrument as a whole and for its individual components. In the present book, we do not treat this subject, however, but limit the discussion to certain other factors which must be considered in instrument design, in order to reduce the effect of vibration on its mechanism.

To reduce the effect of vibration, it is most important that the frequency k of the natural "undamped" oscillations of the mechanism differ as much as possible from the frequency of vibration ω . Were this not the case, vibration of the mechanism might result in forced oscillations of excessive amplitude, which increases the possibility of disruption of the fixed constant kinematic connection between the parts of the mechanism (the latter, as already noted, being a cause of variations in the instrument readings). STAT

In addition, a wide oscillation amplitude in the mechanism is also undesirable since one prerequisite is to reduce the effect of variation in the transmission ratio from the origin of perturbation forces during vibration to the point to which the forces are reduced, as well as to lower the effect of variations in the reduced

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mass of the mechanism. ,

It should be mentioned that, for a large number of instruments, the requirement that the magnitudes of k and ω differ is virtually the only specification made for counteracting dynamic errors (aside from careful balancing of all parts of the mechanism, as far as possible). This is due to the fact that, in the majority of instruments, type I dynamic errors (those due to inertia of the mechanism produced by changes in the quantity to be measured) are usually so small as to be imperceptible, since the possible oscillation frequency of the quantity being measured is usually only a fraction of that of the natural oscillations of the mechanism.

The frequency of the natural oscillations of instrument mechanisms designed for operation under vibration conditions is usually calculated when the instrument is in the design stage; in the final determination of the parameters of the pickup element and the transmitter mechanism of instruments, an effort is always made to assure that $k \neq \omega$.

As an example of such a choice, we may cite the determination of the number of diaphragm cells to be used in instruments for measuring the pressure.

This problem is often encountered in practical experience since an increase in the number of cells in the pickup assembly of an instrument is often used for increasing its total deflection. This makes it possible to reduce the transmission ratio of the mechanism, resulting in a reduction in the total force of friction in the mechanism reduced to the pickup element, and with it the error of friction in the instrument.

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The period T of natural "undamped" oscillations in an instrument mechanism is equal to

$$T = 2\pi \sqrt{\frac{M}{S}},$$

where M is the total mass of the mechanism, reduced to some specific point;

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S is the total elasticity reduced to the same point.

This formula is an approximation, since the presence of some degree of damping of the oscillations (which is always the case in one form or another) changes the period somewhat. It must be noted, however, that the degree of damping in the instruments in question is often small and the resultant change in the period is insignificant.

The elasticity S in this equation comprises the magnitude of the concentrated force to be applied to the point in the mechanism under examination in order for this point to shift by a unit of length. In instruments measuring pressure we usually have at our disposal experimental data revealing the relationship between pressure and the deflection of the pickup element, i.e., we have data on the magnitude of deflection under distributed (but not concentrated) loads. In these cases, in order to determine the magnitude of S , we usually merely multiply the numbers expressing the ratio of pressure to the corresponding deflection of the pickup element, and the effective area of this pickup element. It is obvious that, when using this method, we reduce the elasticity of the cell to its rigid center, i.e., to that point at which the rest of the transmitting mechanism of the instrument is connected.

Note: More accurate results would be obtained if, to the elasticity of the cell (or series thereof) calculated in this manner, there would be added the elasticity of the hair spring, reduced to that point. In many instruments, however, this reduced elasticity of the hair spring is small relative to the elasticity of the pickup element; therefore, the elasticity of the hair spring is usually ignored in this approximated calculation. STAT

The total reduced mass M is the sum of the mass of the pickup element, reduced to the given point, and the reduced mass of the transmitting mechanism.

Let us denote by M_1 the mass of one cell reduced to its moving rigid center. Rigorous calculation of this mass is quite complicated, but it may be regarded as

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being approximately equal to about 40% of the total mass of the cell (since the cell is fixed on one side); this is adequate for purposes of approximate calculation. Further, we denote by M_2 the reduced mass of the transmission mechanism for the case of a single cell, while S_1 is the elasticity of a single cell.

Thus, for a single cell, the natural period of oscillation of the system is approximately equal to

$$T = 2\pi \sqrt{\frac{M_1 + M_2}{S_1}}.$$

If, adhering to the same scale, we enter not one but n identical cells, the transmission ratio to the indicator is diminished by a factor of n . In this situation, the transmission ratio to certain parts of the mechanism also declines by a factor of n and, for certain specific intermediate parts, by somewhat less than a factor of n . Therefore, the reduced mass of certain parts is diminished by a factor of n^2 , and in certain specific parts the reduced mass is diminished by somewhat less than n^2 . In some instances, the decline in the overall transmission ratio of the mechanism will of course permit eliminating certain intermediate parts completely. Overall, the total reduced mass M_{np} of the transmission mechanism will decline by $\frac{n^2}{i^2}$, where i is a number not much greater than unity. Thus,

$$M_{np} = \frac{M_2}{n^2} i^2.$$

The reduced mass M_k of the pickup element (cell) thus increases by a factor of n

$$M_k = M_1 n.$$

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The elasticity S_n of a battery of n cells is smaller by n times than the elasticity of a single cell, since the battery deflects, under the same load, n times as

much as does a single cell. Thus,

$$S_n = \frac{S_1}{n}.$$

Thus, the natural period of oscillation of the mechanism T_n is

$$T_n = 2\pi \sqrt{\frac{M_1 n + \frac{M_2}{n^2} l^2}{\frac{S_1}{n}}} = 2\pi \sqrt{\frac{M_1 n^2 + \frac{M_2}{n} l^2}{S}}$$

and

$$\frac{T_n}{T} = \sqrt{\frac{M_1 n^2 + \frac{M_2}{n} l^2}{M_1 + M_2}}.$$

(4.1)

We see from this equation that, if the reduced mass M_{np} of the transmission mechanism is not very large relative to the mass of the cell M_k (as is the case with certain visual instruments such as altimeters), an increase in the number of cells usually increases the period of oscillation of the system, i.e., reduces its natural frequency. A reduction in the number of cells, however, increases the natural frequency of the system.

In a case in which the reduced mass M_{np} of the transmitting mechanism is several times as large as the reduced mass M_k of the pickup element, any reduction in the number of cells may often increase the oscillation period of the system. This is the usual situation in calculating self-recording devices. In designing these instruments it is usually impossible to obtain a natural frequency of the system greater than the highest presumed fundamental frequency of vibration of the aircraft. Therefore, the oscillation frequency of the system must merely be made smaller than the lowest vibration frequency of the aircraft, capable of being sustained for some length of time in flight (i.e., vibration frequencies of very short duration are ig-

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nored, as for example those occurring when an engine is raced after starting).

If the upper and lower limits of the vibration frequency of an aircraft are known, eq.(4.1) can be used for an approximate advance determination (in designing instruments) of the number of cells required in the given instrument.

In certain diaphragm instruments, used for measuring differential pressures (such as aircraft tachometers), the natural oscillation frequency of the mechanism, calculated in the above manner, differs very little from the vibration frequency of the installation point of the instrument (and in some cases may coincide with certain of the possible vibration frequencies). Here, too, it is often possible to avoid the effect of resonance, if the space within the pressure cell communicates with the pressure source (or with the external atmosphere) through a small nozzle (capillary) installed in the nipple of the instrument. This gives results somewhere between pronounced damping of the system and considerable increase in the elasticity of the pickup element in the case of the usual high vibration frequency (and the resultant increase in the natural vibration frequency of the mechanism), while when the variations in the pressure to be measured are comparatively slow, the elasticity of the system does not change, i.e., it does not increase due to the presence of the nozzle.

Generally speaking, the use of a nozzle should result only in damping the system. This would be the case if the cell contained not air but a liquid which could be considered practically incompressible. However, in view of the compressibility of air, the excess or lack of such air (due to the inability of the air to traverse the nozzle) creates an effect similar to that produced by an extra spring.

A calculation of the approximate magnitude of the resultant damping or increase in the elasticity of the system is simple.

If we were dealing only with damping (i.e., if air were not compressible), the damping coefficient could readily be determined as follows: STAT

In accordance with Poisenille's law, the volume of air passing through the

capillary Q will be

$$Q \approx \frac{\pi r^4 \Delta p}{8 \mu l},$$

where Δp is the pressure drop across the two ends of the capillary;

r is the radius of the capillary;

l is the length of the capillary;

μ is the coefficient of viscosity (measured in kg-sec/cm²). From this, we obtain

$$\Delta p \approx \frac{8 \mu l}{\pi r^4} Q.$$

On the other hand,

$$Q \approx \dot{x} S_{eff},$$

where x is the deflection of the cell;

S_{eff} is the effective area of the cell.

Then,

$$\Delta p \approx \frac{8 \mu l}{\pi r^4} S_{eff} \dot{x}.$$

The damping force F_d is

$$F_d = \Delta p S_{eff} \approx \frac{8 \mu l}{\pi r^4} S_{eff}^2 \dot{x}.$$

Thus, the damping coefficient B [see eqs.(2.8), (3.2), and others] is approximately equal to

$$B \approx \frac{8 \mu l}{\pi r^4} S_{eff}^2 \quad \text{STAT} \quad (4.02)$$

It is obvious that, when the parameters of the capillary are properly selected,

this equation can be used for obtaining a considerable degree of damping (i.e., an $\frac{h}{k}$ ratio in accordance with the reduced values), at which resonance will no longer be dangerous (see Fig.17).

If, for example, a 5-mm capillary is used having a diameter of $d = 2r = 0.4$ mm, and if the effective area of the cell is 8.6 cm^2 , while $\mu = 1.71 \times 10^{-6} \text{ kg-sec-m}^{-2}$ (the viscosity of air at 0°C), we obtain

$$B \approx \frac{8 \cdot 1.71 \cdot 0.5 \cdot 8.6^2}{10^6 \cdot 10^4 \pi \cdot 0.02^4} \text{ kg sec/cm} \approx 0.1 \text{ kg sec/cm},$$

which constitutes a high damping coefficient for the types of mechanisms under consideration, since the reduced mass m_{np} of such mechanisms is quite low (it must be remembered in this connection that, numerically, mass equals weight divided by 981) and that, in order to prevent resonance peaks in the graph in Fig.17, it is sufficient for $\left(\frac{B}{2m_{np}}\right)^2 \geq 0.5 \frac{c}{m_{np}}$, i.e., for $B^2 \geq 2c \cdot m_{np}$, where c is the rigidity of the elastic element in kg/cm .

If, on the other hand, we assume that it is utterly impossible for any air to enter and leave the cell through the nozzle, the supplementary elasticity of the cell obtained in this fashion may be calculated approximately as follows:

The relative variation in the internal volume of the cell, as its deflection changes, is equal approximately to

$$\frac{\Delta V}{V} \approx \frac{x S_{eff}}{V},$$

where V is the volume of the cell with the instrument in the given position;

x is the deflection of the cell in either direction due to vibration.

The approximate magnitude of the relative variation $\frac{\Delta p}{p}$ in absolute pressure p (due to the foregoing), resulting within the cell, is determined from the equation

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$$pV \approx (p + \Delta p)(V + \Delta V).$$

Note: This equation is, of course, not completely accurate, since the process in this case will obviously be more nearly adiabatic; however, for purposes of approximate rough calculation we may make use of it.

From this, we obtain

$$\Delta p V + \Delta V p + \Delta p \cdot \Delta V = 0.$$

Ignoring magnitudes of the second order of smallness (since we consider the amplitude of variation in the magnitude of x to be small), we have

$$\frac{\Delta p}{p} \approx -\frac{\Delta V}{V} = -\frac{x S_{eff}}{V};$$

$$\Delta p \approx -\frac{p S_{eff}}{V} x.$$

From this we find the force of F , reduced to the rigid center of the cell, to be

$$F = \Delta p S_{eff} \approx -\frac{p S_{eff}^2}{V} x, \quad (4.3)$$

where p is the absolute pressure in the cell at a given reading of the instrument.

Thus, the magnitude $\frac{p S_{eff}^2}{V}$ is the additional elasticity of the cell to be added to its natural elasticity. It is quite obvious that the additional elasticity calculated by means of this last equation may prove quite considerable, at times even exceeding the natural elasticity of the cell.

For example, if we again consider the effective area of the cell to be 8.6 cm^2 , its internal volume 7.4 cm^3 and its pressure $p \approx 1 \text{ kg/cm}^2$, we obtain

$$\frac{p S_{eff}^2}{V} = \frac{1.8,6^2}{7,4} \approx 10 \text{ kg cm}.$$

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Thus, the nozzle discussed above always causes a reduction in the vibration effect, regardless of how we look at it, i.e., whether we consider that it damps the

system or that it increases its elasticity (in the case of high-frequency vibration).

Section 2. Effect of Coulomb Friction in Steady-State Forced Vibrations

In the preceding Sections, when deriving our equations for forced oscillations of the mechanisms, we ignored the Coulomb friction for the sake of simplicity (although it is true that Coulomb friction is usually not large in instruments). In the present Section, we will show how Coulomb friction affects the forced oscillations of any body (as in the cases examined above) possessing a force of recovery proportional to the first power of its deviation from the center position, in a case in which the perturbing force takes on the form of a harmonic function. Although this conclusion is not applicable to all instruments, we draw it here since it is at least applicable to some cases. We will also indicate the conditions under which it cannot be used, and the reasons for this. The general idea of this conclusion is known from the literature (Bibl.5); however, we will give a fuller and more detailed presentation of the subject.

For the sake of simplicity, let us first analyze a case in which damping is absent. In this case, we have the equation

$$m\ddot{x} + Sx + F_{\text{tr}} = A \cos \omega t, \quad (4.4)$$

where m is the mass of the body;

S is the specific force of recovery (elasticity);

x is the displacement of the body;

$A \cos \omega t$ is the perturbing force (A being a constant).

Let us examine only a case of steady-state oscillation, in which a body oscillates in complete symmetry first to one and then to the other side of its center position. Let us examine the motion from one extreme position of the body to the other extreme position, i.e., from $+x_0$ to $-x_0$, where x_0 is the amplitude of the steady-state vibrations. In other words, let us examine the half-cycle of oscillation in

which the body is displaced to one side. Let us assume that the direction of motion during this period of time does not change at all, so that the sign of the force of friction F_{Tp} also remains the same.

This will give us the equation

$$m\ddot{x} + Sx - F_{Tp} = A \cos \omega t. \quad (4.5)$$

As, in the course of motion from $+x_0$ to $-x_0$, the force of friction F_{Tp} is positive, eq.(4.5) is written as follows if the correct sign be written in front of all its terms:

$$-m\ddot{x} - Sx + F_{Tp} = -A \cos \omega t.$$

The solution of eq.(4.5) takes on the form

$$x = D \cos \omega t + \frac{F_{Tp}}{S} + c_1 \sin kt + c_2 \cos kt,$$

where $D = \frac{A}{S - m\omega^2}$ is the amplitude of the forced oscillations in the absence of friction;

$$k = \sqrt{\frac{S}{m}}.$$

Let us assume the body to be in a position determined by the magnitude $+x_0$ at a given instant of time t_1 . Then, the solution of eq.(4.5) may be written in the following form:

$$x = D \cos \omega t + \frac{F_{Tp}}{S} + c_3 \sin k(t - t_1) + c_4 \cos k(t - t_1). \quad (4.6)$$

In this expression, only the integration constants c_3 and c_4 vary, which have not yet been determined. Let us examine only the case in which a body is in constant motion during a complete half-cycle, from position $+x_0$ to $-x_0$, i.e., the body

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is never in the position $-x_0$ before it begins to return. Below, we will derive the conditions under which this requirement is met [see the inequality (4.19)].

Thus the body reaches a position corresponding to the magnitude $-x_0$ at the instant

$$t_1 + \frac{T}{2} = t_1 + \frac{\pi}{\omega}.$$

Thus, in a case of steady-state oscillations, $x_0 = -(-x_0)$. In addition, in the extreme positions (when $x = +x_0$ and $x = -x_0$), i.e., at the time t_1 and $t_1 + \frac{\pi}{\omega}$, the velocity $\dot{x}$ equals zero. This yields three equations

$$D \cos \omega t_1 + \frac{F_{rp}}{S} + c_4 = - \left[D \cos (\omega t_1 + \pi) + \frac{F_{rp}}{S} + c_3 \sin \frac{k}{\omega} \pi + c_4 \cos \frac{k}{\omega} \pi \right]. \quad (4.7)$$

$$-D\omega \sin \omega t_1 + c_3 k = 0. \quad (4.8)$$

$$-D\omega \sin (\omega t_1 + \pi) + c_3 k \cos \frac{k}{\omega} \pi - c_4 k \sin \frac{k}{\omega} \pi = 0. \quad (4.9)$$

Since

$$\begin{aligned} \sin (\omega t_1 + \pi) &= -\sin \omega t_1; \\ \cos (\omega t_1 + \pi) &= -\cos \omega t_1, \end{aligned}$$

equations (4.7) and (4.9) take on the form

$$D \cos \omega t_1 + \frac{F_{rp}}{S} + c_4 = D \cos \omega t_1 - \frac{F_{rp}}{S} - c_3 \sin \frac{k}{\omega} \pi - c_4 \cos \frac{k}{\omega} \pi,$$

or

$$c_3 \sin \frac{k}{\omega} \pi + c_4 \left(1 + \cos \frac{k}{\omega} \pi \right) = -2 \frac{F_{rp}}{S}; \quad (4.7a)$$

$$D\omega \sin \omega t_1 + c_3 k \cos \frac{k}{\omega} \pi - c_4 k \sin \frac{k}{\omega} \pi = 0. \quad (4.9a) \text{ STAT}$$

Three unknowns, t_1 , c_3 , and c_4 , may be determined from eqs. (4.8), (4.7a), and

(4.9a).

From eqs.(4.8) and (4.9a), we derive

$$c_3 \left(1 + \cos \frac{k}{\omega} \pi\right) - c_4 \sin \frac{k}{\omega} \pi = 0.$$

From this equation and from eq.(4.7a) we determine c_3 and c_4

$$\begin{aligned} c_3 &= - \frac{2 \frac{F_{TP}}{S} \sin \frac{k}{\omega} \pi}{\left(1 + \cos \frac{k}{\omega} \pi\right)^2 + \sin^2 \frac{k}{\omega} \pi} = - \frac{2 \frac{F_{TP}}{S} \sin \frac{k}{\omega} \pi}{2 + 2 \cos \frac{k}{\omega} \pi} = \\ &= - \frac{\frac{F_{TP}}{S} 2 \sin \left(\frac{k}{\omega} \frac{\pi}{2}\right) \cos \left(\frac{k}{\omega} \frac{\pi}{2}\right)}{2 \cos^2 \left(\frac{k}{\omega} \frac{\pi}{2}\right)} = - \frac{F_{TP}}{S} \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2}\right); \end{aligned} \quad (4.10)$$

$$c_4 = - \frac{2 \frac{F_{TP}}{S} \left(1 + \cos \frac{k}{\omega} \pi\right)}{\left(1 + \cos \frac{k}{\omega} \pi\right)^2 + \sin^2 \frac{k}{\omega} \pi} = - \frac{2 \frac{F_{TP}}{S} \left(1 + \cos \frac{k}{\omega} \pi\right)}{2 + 2 \cos \frac{k}{\omega} \pi} = - \frac{F_{TP}}{S}. \quad (4.11)$$

In this manner, eq.(4.8) yields

$$\sin \omega t_1 = \frac{c_3 k}{D \omega} = - \frac{F_{TP} k}{D \omega S} \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2}\right). \quad (4.12)$$

And from this

$$\begin{aligned} x_0 &= D \cos \omega t_1 + \frac{F_{TP}}{S} + c_4 = D \cos \omega t_1 + \frac{F_{TP}}{S} - \frac{F_{TP}}{S} = \\ &= D \cos \omega t_1 = D \sqrt{1 - \sin^2 \omega t_1} = \sqrt{D^2 - D^2 \sin^2 \omega t_1} = \\ &= \sqrt{D^2 - \frac{k^2 F_{TP}^2}{\omega^2 S^2} \operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2}\right)} = \\ &= \sqrt{\frac{A^2}{(S - m \omega^2)^2} - \frac{F_{TP}^2}{m S \omega^2} \operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2}\right)}. \end{aligned} \quad \begin{array}{l} \text{STAT} \\ (4.13) \end{array}$$

The ωt_1 phase shift is determined from eqs.(4.12).

Equation (4.12) itself yields one limitation of the applicability of our conclusions, since its conditions can obviously be satisfied only if

$$\left| \frac{F_{TP} k}{DS\omega} \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right) \right| < 1,$$

or, in other words, it is required that

$$\left| \frac{F_{TP}}{A} \frac{k^2 - \omega^2}{k\omega} \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right) \right| < 1. \quad (4.14)$$

However, if this condition is not satisfied, the entire deduction advanced above is inapplicable.

It is of interest to note the following: In accordance with eq.(4.13), the amplitude of oscillations in the presence of Coulomb friction is

$$x_0 = \frac{A}{S - m\omega^2} \cos \omega t_1,$$

where ωt_1 is the phase shift between the oscillations of the perturbing force and the forced oscillations of the mechanism. However, in the case of ordinary damping, proportional to the first power of velocity $\dot{x}$, we obtain the following magnitude for the oscillation amplitude [see eq.(2.16)]:

$$\begin{aligned} x_0 &= \frac{A}{\sqrt{(S - m\omega^2)^2 + 4m^2\omega^2 h^2}} = \frac{A}{(S - m\omega^2) \sqrt{1 + \frac{4m^2\omega^2 h^2}{(S - m\omega^2)^2}}} \\ &= \frac{A}{S - m\omega^2} \frac{1}{\sqrt{1 + \frac{4\omega^2 h^2}{(k^2 - \omega^2)^2}}} = \frac{A}{S - m\omega^2} \frac{1}{\sqrt{1 + \operatorname{tg}^2 \delta}} = \frac{A}{S - m\omega^2} \cos \delta, \end{aligned}$$

since $\frac{2\omega h}{k^2 - \omega^2} = \operatorname{tg} \delta$, where δ is the phase shift between the oscillations of the perturbing force and the oscillations of the mechanism in the presence of a damping

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factor h .

Thus, if the reduction in amplitude x_0 of the forced oscillations, due to damping or Coulomb friction, is expressed as a function of the resultant phase shift, these expressions will be completely identical.

From this, we may derive the expression for the "damping equivalent of Coulomb friction" in a case in which the perturbing force is harmonic in nature; if this "equivalent" is denoted by h_1 , then, by equating the tangents of the corresponding angles of phase shift, we obtain

$$\begin{aligned} \frac{2\omega h_1}{k^2 - \omega^2} &= \frac{F_{Tp} k (S - m\omega^2) \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right)}{\sqrt{A^2 \omega^2 S^2 - F_{Tp}^2 k^2 (S - m\omega^2)^2 \operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2} \right)}} = \\ &= \frac{F_{Tp} (k^2 - \omega^2) \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right)}{\sqrt{A^2 k^2 \omega^2 - F_{Tp}^2 (k^2 - \omega^2)^2 \operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2} \right)}}, \end{aligned}$$

from which we derive

$$h_1 = \frac{F_{Tp} (k^2 - \omega^2) \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right)}{2\omega \sqrt{A^2 k^2 \omega^2 - F_{Tp}^2 (k^2 - \omega^2)^2 \operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2} \right)}} \quad (4.15)$$

It should be mentioned that the magnitude of the resultant "equivalent" depends on the relation between the magnitude of the force of friction F_{Tp} , and the magnitude of the amplitude of the perturbing force A , while the magnitude h in the case of ordinary damping does not depend on the magnitude A . Thus, when the mechanism is well balanced, the Coulomb friction in the mechanism may be equivalent to a high power of the damping, while if the mechanism presents marked inequilibrium, the same reduced force of friction may prove equivalent to a considerably lower damping. STAT

Here one remark has to be made with respect to the matter of how to employ

eq.(4.12). Let us note first that, in accordance with this equation, the magnitude of $\sin \omega t_1$ is always positive if the $\frac{k}{\omega}$ ratio is within 0 and 2 as limits. It is true that, if $k < \omega$, the magnitude of $\tan \left(\frac{k}{\omega} \frac{\pi}{2} \right)$ is positive and D is negative, while when $\omega < k < 2\omega$, the value $\tan \left(\frac{k}{\omega} \frac{\pi}{2} \right)$ is negative and D is positive.

In a case in which $k > 2\omega$, the value of $\sin \omega t_1$ may be either positive or negative, depending on the magnitude of the $\frac{k}{\omega}$ ratio. As may be seen from the following, it is hardly possible in such cases to satisfy all the conditions under which the deduction made is feasible.

For this reason, we will not examine cases in which $k > 2\omega$.

Thus, let us examine eq.(4.12) for cases in which the $\frac{k}{\omega}$ ratio lies within the limits of 0 and 2, so that $\sin \omega t_1$ is positive.

For each particular positive value of $\sin \omega t_1$, there are two values of the ωt_1 angle, namely one in which $0 < \omega t_1 < \frac{\pi}{2}$, and another in which $\frac{\pi}{2} < \omega t_1 < \pi$, with the sum of these two possible values for the ωt_1 angle being π . It is, therefore, necessary to establish which of these two values must be used in either instance.

Here the following rule may be applied: If $k < \omega$, then the solution in which $\pi > \omega t_1 > \frac{\pi}{2}$ is to be used; if $\omega < k < 2\omega$, the solution in which $\omega t_1 < \frac{\pi}{2}$ must be used.

Among the further considerations confirming this rule, the following may be cited: If $F_{Tp} = 0$, the phase shift is equal to zero ($\omega t_1 = 0$) when $k > \omega$. If, after this step, a very slight friction is introduced and then gradually increased, the value of ωt_1 will steadily increase in difference from that of $\omega t_1 = 0$ (specifically, starting at that value). Therefore, the value of ωt_1 that is smaller than $\frac{\pi}{2}$ is used here.

Conversely, if $F_{Tp} = 0$ and $k < \omega$, then $\omega t_1 = \pi$ and, with a gradual increase in friction, the value of ωt_1 will gradually deviate from this value (i.e., starting specifically with $\omega t_1 = \pi$). In addition, only when the above rule is observed will the value of $+x_0 = D \cos \omega t_1$ be positive both when $k < \omega$ and when $\omega < k < 2\omega$, as is

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required by the above deductions.

Substituting the obtained values for the integration constants c_3 and c_4 [see eqs.(4.10) and (4.11)], we obtain the following in eq.(4.6):

$$\left. \begin{aligned} x &= D \cos \omega t + \frac{F_{Tp}}{S} - \frac{F_{Tp}}{S} \frac{\cos k \left(t - t_1 - \frac{\pi}{2\omega} \right)}{\cos k \frac{\pi}{2\omega}}; \\ \dot{x} &= -D\omega \sin \omega t + \frac{F_{Tp}}{S} k \frac{\sin k \left(t - t_1 - \frac{\pi}{2\omega} \right)}{\cos k \frac{\pi}{2\omega}}; \\ \ddot{x} &= -D\omega^2 \cos \omega t + \frac{F_{Tp}}{S} k^2 \frac{\cos k \left(t - t_1 - \frac{\pi}{2\omega} \right)}{\cos k \frac{\pi}{2\omega}}. \end{aligned} \right\} \quad (4.16)$$

At the beginning of the first swing we have

$$\left. \begin{aligned} \dot{x}_{01} &= -D\omega \sin \omega t_1 - \frac{F_{Tp}}{S} k \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right) = 0; \\ \ddot{x}_{01} &= -D\omega^2 \cos \omega t_1 + \frac{F_{Tp}}{S} k^2, \end{aligned} \right\} \quad (4.17)$$

and at the end of the first swing

$$\left. \begin{aligned} \dot{x}_{k1} &= D\omega \sin \omega t_1 + \frac{F_{Tp}}{S} k \operatorname{tg} \left(\frac{k}{\omega} \frac{\pi}{2} \right) = 0; \\ \ddot{x}_{k1} &= D\omega^2 \cos \omega t_1 + \frac{F_{Tp}}{S} k^2. \end{aligned} \right\} \quad (4.18)$$

Note: Since, the beginning of the second swing, we must have an acceleration of

$$\ddot{x}_{02} = D\omega \cos \omega t_1 - \frac{F_{Tp}}{S} k^2, \quad (4.17a) \text{ STAT}$$

the acceleration suddenly changes by $2 \frac{F_{Tp}}{S} k^2 = 2 \frac{F_{Tp}}{m}$, at the instant when

the body reaches the position $-x_0$, resulting from the fact that the sign for the force of friction changes at this instant.

We indicated above that we are here discussing only the case in which the body does not remain in the $-x_0$ position for any length of time, i.e., only a case in which, after reaching the $-x_0$ position, the body immediately begins to move in the reverse direction. In order for this to occur, the following condition is required [see eq.(4.17a)]:

$$\begin{aligned} D\omega^2 \cos \omega t_1 &\geq \frac{F_{Tp}}{S} k^2; \\ D^2\omega^4 \cos^2 \omega t_1 &\geq \frac{F_{Tp}^2}{S^2} k^4; \\ D^2\omega^4 \left[1 - \frac{F_{Tp}^2 k^2}{D^2\omega^2 S^2} \operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2} \right) \right] &\geq \frac{F_{Tp}^2}{S^2} k^4, \end{aligned}$$

from which we obtain

$$\left(\frac{F_{Tp} k}{D\omega S} \right)^2 \left[\operatorname{tg}^2 \left(\frac{k}{\omega} \frac{\pi}{2} \right) + \frac{k^2}{\omega^2} \right] \leq 1. \quad (4.19)$$

This is the fundamental requirement to be met by the above conclusion in order to be feasible, since it includes the condition (4.14). Satisfaction of eq.(4.19) necessarily includes satisfaction of eq.(4.14).

In addition, the method of allowing for Coulomb friction in forced vibrations is applicable only if the direction of motion does not vary during the period of time embraced by the half-cycle while the body moves from the $+x_0$ to the $-x_0$ period; it is only if this condition is satisfied that the sign of F_{Tp} which is basic to our conclusion, does not vary during this period of time. However, certain calculations for purposes of orientation have shown that adherence to this latter condition is always related to satisfying condition (4.19). This latter condition is also confirmed by certain concepts, developed by graphic analysis.

Let us plot graphs for the acceleration and velocities of bodies in motion from the point $+x_0$ to the point $-x_0$. Since, in accordance with eqs.(4.16), the accelerations and velocities are expressed as the sum of two sinusoids, these sinusoids are plotted on graphs. The algebraic sums of the ordinates of these sinusoids gives the magnitude of the accelerations or the velocities.

Let us begin by examining the case in which $k < \omega$. Then $D < 0$, and the sinusoid $-D\omega^2 \cos \omega t$ takes approximately the form of curve I in Fig.22. Let us take some specimen position for point t_1 corresponding to $\frac{\pi}{2} < \omega t_1 < \pi$.

The sinusoid $\frac{F_{Tp}}{S} k^2 \cdot \cos k \left(t - t_1 - \frac{\pi}{2\omega} \right)$ takes approximately the form of curve II. Let us also plot the broken line III, constituting the mirror image of the curve II relative to the abscissa. Then the difference between the ordinates of curve I and curve III constitutes the acceleration of the body. Here we see that,

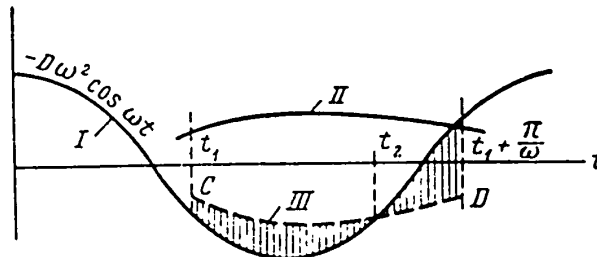


Fig.22 - Effect of Dry Friction in Steady-State Forced Vibrations. Specimen Graph of Accelerations for the Case in which $k < \omega$.

from the instant t_1 to the instant t_2 , accelerations are negative while, from the instant t_2 to the instant $t_1 + \frac{\pi}{\omega}$, the accelerations are positive.

If an analogous graph is plotted for the accelerations (Fig.23), the sinusoid $-D\omega \sin \omega t$ takes approximately the form of curve I, while the sinusoid

$$\frac{F_{Tp}}{S} \frac{k}{\cos \left(\frac{k}{\omega} - \frac{\pi}{2} \right)} \sin k \left(t - t_1 - \frac{\pi}{2\omega} \right) -$$

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takes approximately the form of curve II. Curve III will be the mirror image of the curve II, and the difference between the ordinates in curves I and III will constitute the velocity of the body.

The graphs in Figs.22 and 23 show that the velocities will always be negative

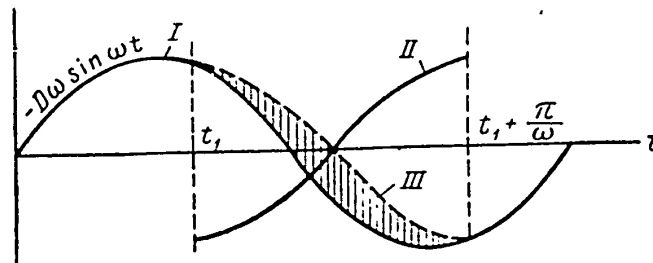


Fig.23 - Effect of Dry Friction in Steady-State Forced Vibrations.

Specimen graph of velocities in the case in which $k < \omega$

over the entire period from t_1 to $t_1 + \frac{\pi}{\omega}$.

Let us now proceed to consider a case in which $\omega < k < 2\omega$. In this case, $D < 0$ and the sinusoid $-D\omega^2 \cos \omega t$ takes approximately the form of the curve I in Fig.24.

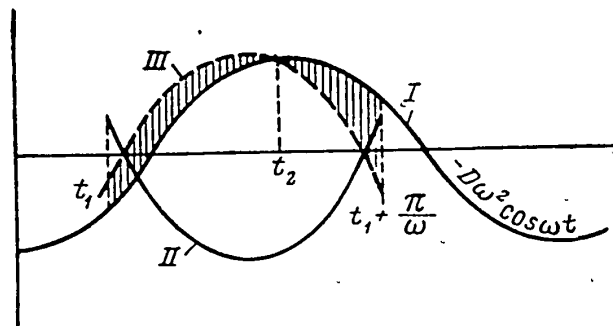


Fig.24 - Effect of Dry Friction in Steady-State Forced Vibration.

Specimen graph of velocities in the case in which $\omega < k < 2\omega$

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Let us take some point t_1 , corresponding to $\omega t_1 < \frac{\pi}{2}$. The sinusoid

$$\frac{F_{TP}}{S} \frac{k^2}{\cos\left(\frac{k}{\omega} \frac{\pi}{2}\right)} \cos k\left(t - t_1 - \frac{\pi}{2\omega}\right)$$

takes approximately the form of curve II since, in this case, $\cos\left(\frac{k}{\omega} \frac{\pi}{2}\right) < 0$. The broken curve III is a mirror image of curve II. The difference in the ordinates of curve I and curve III represents the acceleration of the body. Here again, the accelerations are negative from the instant t_1 to t_2 , while from the instant t_2 to $t_1 + \frac{\pi}{\omega}$ they are positive.

Let us plot a graph of velocities (Fig.25) analogous to the graph in Fig.23. On this graph, the difference in the ordinates of curves I and III represent the

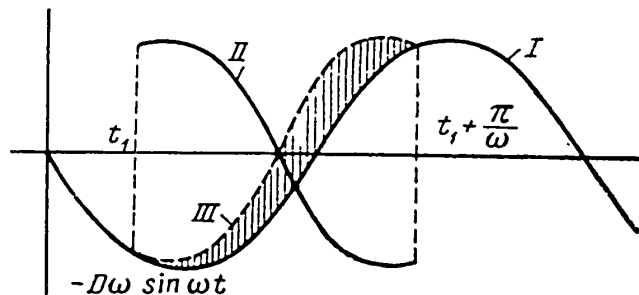


Fig.25 - Effect of Dry Friction in Steady-State Forced Vibration.

Specimen Graph of Velocities in case in which $\omega < k < 2\omega$

velocities of the body. From the graphs in Figs.24 and 25 we again find that the velocities will always be negative for the entire duration from t_1 to $t_1 + \frac{\pi}{\omega}$.

Apparently, however, the picture will be entirely different when $k > 2\omega$; analogous graphs reveal that, in this situation, the velocities will no longer have a constant direction in the segment from t_1 to $t_1 + \frac{\pi}{\omega}$. Therefore, we conclude that the deduction with which we have been working does not extend to this case.

With respect to cases in which $k < 2\omega$, when the condition (4.19) is met, this

apparently meets (for the most part) the requirement that the direction of the velocity remains constant in the segment from t_1 to $t_1 + \frac{\pi}{\omega}$. However, when $\omega < k < 2\omega$, it is more difficult to satisfy the condition (4.19) than when $k < \omega$, in view of the fact that the conclusion under consideration is chiefly applicable to $k < \omega$.

In cases in which damping is present [i.e., the term $B\dot{x}$ in eq.(4.5), where B is the specific force of damping], analogous reasoning is possible. In place of eq. (4.5) we will have the expression

$$m\ddot{x} + B\dot{x} + Sx - F_{TP} = A \cos \omega t. \quad (4.20)$$

The solution of this equation may be written in the following form:

$$x = \frac{A}{S} \lambda \cos(\omega t - \delta) + \frac{F_{TP}}{S} + e^{-h(t-t_1)} [c_3 \sin \sqrt{k^2 - h^2}(t - t_1) + c_4 \cos \sqrt{k^2 - h^2}(t - t_1)], \quad (4.21)$$

where λ is the coefficient of dynamic susceptibility;

δ is the phase shift that would occur if the only damping taking place were $B\dot{x}$ and no Coulomb friction existed (the form of the quantities λ and δ is already known);

$$h = \frac{B}{2m}.$$

As before, we have three equations: $+x_0 = -(-x_0)$ and, in the extreme positions (i.e., at the instants $t = t_1$ and $t = t_1 + \frac{\pi}{\omega}$), velocity $\dot{x} = 0$. Let us write the derivative $\dot{x}$ in its general form

$$\begin{aligned} \dot{x} = & -\frac{A}{S} \lambda \omega \sin(\omega t - \delta) + e^{-h(t-t_1)} \{ -h [c_3 \sin \sqrt{k^2 - h^2}(t - t_1) + \\ & + c_4 \cos \sqrt{k^2 - h^2}(t - t_1)] + \sqrt{k^2 - h^2} [c_3 \cos \sqrt{k^2 - h^2}(t - t_1) - \\ & - c_4 \sin \sqrt{k^2 - h^2}(t - t_1)] \} \end{aligned} \quad \text{STAT} \quad (4.22)$$

Thus, we have three equations analogous to eqs.(4.7a), (4.8), and (4.9a)

$$c_3 n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + c_4 \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) = -2 \frac{F_{TP}}{S}; \quad (4.23)$$

$$-\frac{A}{S} \lambda \omega \sin(\omega t_1 - \delta) - h c_4 + \sqrt{k^2 - h^2} c_3 = 0; \quad (4.24)$$

$$\begin{aligned} & \frac{A}{S} \lambda \omega \sin(\omega t_1 - \delta) - h n \left[c_3 \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + c_4 \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right] + \\ & + \sqrt{k^2 - h^2} n \left[c_3 \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi - c_4 \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right] = 0, \end{aligned} \quad (4.25)$$

where

$$n = e^{-\frac{h}{\omega} \pi}.$$

These three equations enable us to determine the constants c_3 , c_4 and t_1 .

By adding eqs.(4.24) and (4.25), we obtain

$$\begin{aligned} & -h \left[c_4 \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) + c_3 n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right] + \\ & + \sqrt{k^2 - h^2} \left[c_3 \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - c_4 n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right] = 0. \end{aligned} \quad (4.26)$$

From eqs.(4.23) and (4.26) we determine c_3 and c_4

$$\begin{aligned} c_3 &= -2 \frac{F_{TP}}{S} \frac{\left[n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]}{\left[\left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)^2 + \left(n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)^2 \right]} = \\ &= -2 \frac{F_{TP}}{S} \frac{\left[n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]}{\left[1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}; \end{aligned} \quad (4.27)$$

$$c_4 = -2 \frac{F_{TP}}{S} \frac{\left[\left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - \frac{h}{\sqrt{k^2 - h^2}} n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}{\left[1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}. \quad (4.28)$$

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From eq.(4.24) we have

$$\begin{aligned}
 \sin(\omega t_1 - \delta) &= \frac{S(\sqrt{k^2 - h^2} c_1 - h c_2)}{A \omega} \\
 &= - \frac{2 F_{\text{TP}} n \left(\sqrt{k^2 - h^2} + \frac{h^2}{\sqrt{k^2 - h^2}} \right) \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{A \lambda \omega \left(1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)} \\
 &= - \frac{2 F_{\text{TP}} n k^2 \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{A \lambda \omega \sqrt{k^2 - h^2} \left(1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)}. \quad (4.29)
 \end{aligned}$$

From this equation we are in a position to determine the phase shift $\omega t_1 - \delta$.

Further,

$$\begin{aligned}
 x_0 &= \frac{A}{S} \lambda \cos(\omega t_1 - \delta) + \frac{F_{\text{TP}}}{S} + c_4 = \\
 &= \frac{A}{S} \sqrt{\lambda^2 - \frac{4 F_{\text{TP}}^2 n^2 k^4 \sin^2 \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{A^2 \lambda^2 \omega^2 (k^2 - h^2) \left(1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)^2}} + \\
 &+ \frac{F_{\text{TP}}}{S} \left\{ 1 - \frac{2 \left[\left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - \frac{h}{\sqrt{k^2 - h^2}} n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}{\left(1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)} \right\} = \\
 &= \sqrt{\left(\frac{A}{S} \lambda \right)^2 - \frac{k^2}{k^2 - h^2} \frac{F_{\text{TP}}^2 n^2}{m \lambda^2 S \omega^2} \frac{4 \sin^2 \left(\frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)}{\left[1 + n^2 + 2n \cos \left(\frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]^2}} + \\
 &+ \frac{F_{\text{TP}}}{S} \frac{\left[n^2 - 1 + 2 \frac{h}{\sqrt{k^2 - h^2}} n \sin \left(\frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]}{\left[1 + n^2 + 2n \cos \left(\frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]}. \quad (4.30) \quad \text{STAT}
 \end{aligned}$$

In determining the phase shift in accordance with eq.(4.29) it is no longer

necessary to select, on each occasion, one or another of the two possible values of ωt_1 as a function of the relationship of the frequencies of the natural and forced vibrations of the system, as we had done in the case in which damping was absent. Actually, here the phase shift caused by factors other than friction (such as inertia and damping) has already been covered in the value of δ ; therefore, if at the outset a low friction value is introduced which is then increased, the value of ωt_1 will gradually come to differ from the value of $\omega t_1 = \delta$ (specifically, starting at this value), at any ratio of the natural to the forced vibration frequencies.

Thus, in this case it is always necessary to use the value of the quantity $\omega t_1 - \delta$ at which $|\omega t_1 - \delta| < \frac{\pi}{2}$; here, if $\sqrt{k^2 - h^2} < \omega$, the value of $\sin(\omega t_1 - \delta)$ (and therefore the value of $\omega t_1 - \delta$) will now be negative.

Here we also have a case limiting the applicability of the entire above derivation, in a manner analogous to condition (4.14). In actuality, eq.(4.29) is obviously valid only if

$$\left| \frac{2F_{TP} n k^2 \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{A \lambda \omega \sqrt{k^2 - h^2} \left(1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)} \right| < 1. \quad (4.31)$$

On the basis of eq.(4.22), we have

$$\begin{aligned} \ddot{x} = & -\frac{A}{S} \lambda \omega^2 \cos(\omega t_1 - \delta) + \\ & + e^{-h(t-t_1)} \{ (2h^2 - k^2) [c_3 \sin \sqrt{k^2 - h^2}(t - t_1) + c_4 \cos \sqrt{k^2 - h^2}(t - t_1)] - \\ & - 2h \sqrt{k^2 - h^2} [c_3 \cos \sqrt{k^2 - h^2}(t - t_1) - c_4 \sin \sqrt{k^2 - h^2}(t - t_1)] \}. \end{aligned} \quad (4.32)$$

On this basis, at the start of the first swing (when $t - t_1 = 0$), substitution of the values of the integration constants c_3 and c_4 from eqs.(4.27) and (4.28) in ^{STAT} eq.(4.32) will result in an acceleration $\ddot{x}_{01}$, which equals

$$\begin{aligned}
\ddot{x}_{01} = & -\frac{A}{S} \lambda \omega^2 \cos(\omega t_1 - \delta) - \frac{2F_{TP}}{S} \times \\
& \times \left\{ (2h^2 - k^2) \frac{\left[\left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - \frac{h}{\sqrt{k^2 - h^2}} n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} - \right. \\
& \left. - 2h \sqrt{k^2 - h^2} \frac{\left[n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \right\} = \\
& = \frac{2F_{TP}}{S} k^2 \frac{1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} - \\
& - \frac{A}{S} \lambda \omega^2 \cos(\omega t_1 - \delta) \quad (4.33)
\end{aligned}$$

At the end of the first swing (when $t - t_1 = \frac{\pi}{\omega}$), we have an acceleration of x_{k1} , equal to

$$\begin{aligned}
\ddot{x}_{k1} = & \frac{A}{S} \lambda \omega^2 \cos(\omega t_1 - \delta) - \frac{2F_{TP}}{S} \times \\
& \times n \left(\left((2h^2 - k^2) \frac{\left[n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \times \right. \right. \\
& \times \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{\left[\left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - \frac{h}{\sqrt{k^2 - h^2}} n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \times \\
& \left. \left. \times \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - 2h \sqrt{k^2 - h^2} \times \right)
\end{aligned}$$

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$$\begin{aligned}
& \times \left[\frac{n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi - \right. \\
& \quad \left. - \frac{\left[\left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right) - \frac{h}{\sqrt{k^2 - h^2}} n \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \times \right. \\
& \quad \left. \times \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right] \left. \right) = \frac{A}{S} \lambda \omega^2 \cos (\omega t_1 - \delta) - \\
& - \frac{2F_{TP}}{S} n \left\{ (2h^2 - k^2) \frac{\left[\frac{h}{\sqrt{k^2 - h^2}} \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi + n \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} - \right. \\
& \quad \left. - 2h \sqrt{k^2 - h^2} \frac{\left[\frac{h}{\sqrt{k^2 - h^2}} \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi - \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi + \frac{h}{\sqrt{k^2 - h^2}} n \right]}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \right\} = \\
& = \frac{A}{S} \lambda \omega^2 \cos (\omega t_1 - \delta) + \\
& + \frac{2F_{TP}}{S} k^2 n \frac{n + \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi - \frac{h}{\sqrt{k^2 - h^2}} \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \quad (4.34)
\end{aligned}$$

At the start of the second (reverse) swing, we should have an acceleration $\ddot{x}_{o2}$, with a value opposite to that of the acceleration $\ddot{x}_{o1}$ (since we are assuming symmetrical motion), i.e., an acceleration [see eq.(4.33)] equal to

$$\begin{aligned}
\ddot{x}_{o2} &= \frac{A}{S} \lambda \omega^2 \cos (\omega t_1 - \delta) - \\
& - \frac{2F_{TP}}{S} k^2 \frac{1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi + n \frac{h}{\sqrt{k^2 - h^2}} \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi}{1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi} \quad \text{STAT} \\
& \quad (4.35)
\end{aligned}$$

As expected, the acceleration at the instant of cessation of motion suddenly declines by

$$\begin{aligned} x_{k1} - \ddot{x}_{02} &= \frac{2F_{TP}}{S} k^2 \frac{1+n^2+2n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi}{1+n^2+2n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi} = \\ &= 2 \frac{F_{TP}}{S} k^2 = 2 \frac{F_{TP}}{m}. \end{aligned}$$

By making use of eq.(4.35) or eq.(4.33), it is possible to obtain a condition analogous to that described by eq.(4.19); if this condition is satisfied, the body will not remain in its extreme positions for any length of time. What is required for this is that

$$\begin{aligned} \frac{A}{S} \lambda \omega^2 \cos(\omega t_1 - \delta) &\geq \frac{2F_{TP}}{S} \times \\ &\times k^2 \frac{1+n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi + n \frac{h}{\sqrt{k^2-h^2}} \sin \frac{\sqrt{k^2-h^2}}{\omega} \pi}{1+n^2+2n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi}. \end{aligned}$$

From this, by using eq.(4.29), we obtain

$$\begin{aligned} \left(\frac{A}{S} \lambda \omega^2 \right)^2 &\left\{ 1 - \left[\frac{2F_{TP} n k^2 \sin \frac{\sqrt{k^2-h^2}}{\omega} \pi}{A \lambda \omega \sqrt{k^2-h^2} \left(1+n^2+2n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi \right)} \right]^2 \right\} \geq \\ &\geq \left[\frac{2F_{TP}}{S} k^2 \frac{1+n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi + n \frac{h}{\sqrt{k^2-h^2}} \sin \frac{\sqrt{k^2-h^2}}{\omega} \pi}{1+n^2+2n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi} \right]^2. \end{aligned}$$

from which we derive

$$\left(\frac{2F_{TP} k}{A \lambda \omega} \right)^2 \left\{ \frac{n^2 \frac{k^2}{k^2-h^2} \sin^2 \frac{\sqrt{k^2-h^2}}{\omega} \pi}{\left(1+n^2+2n \cos \frac{\sqrt{k^2-h^2}}{\omega} \pi \right)^2} + \right.$$

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$$\frac{\frac{k^2}{\omega^2} \left(1 + n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi + n \frac{h}{\sqrt{k^2 - h^2}} \sin \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)^2}{\left(1 + n^2 + 2n \cos \frac{\sqrt{k^2 - h^2}}{\omega} \pi \right)^2} \leq 1. \quad (4.36)$$

Here we also see that satisfaction of condition (4.36) also satisfies condition (4.31).

Thus it will be seen that, where damping is absent and where it is present, Coulomb friction, at the beginning of each swing (as the direction of motion changes) will generate natural vibrations of the type discussed in preceding Sections.

Section 3. Balancing Instrument Mechanisms, Design of the Hair Spring, and Damping of Oscillations of Instrument Mechanisms

In view of the fact that, in many cases, one of the most likely sources of changes in instrument readings during vibration is a disruption of the constant determinate kinematic connection between the parts of the instrument mechanism, certain considerations must be made to ensure that this kinematic connection will remain constant:

1. It is essential to keep the hair spring under adequate tension. The degree of tension required will be specified below in Chapter X [eq.(10.10)].
2. If a cranked connection in the transmission mechanism is used to balance the pickup element of the instrument (as occurs, for example, in altimeters, climb indicators, etc.), it is advisable to bring about this balance by means of the diagram depicted in Fig.26, which is used in altimeters, and not in that of Fig.27, used in designing other instruments.

In altimeters, diaphragm cells connected by means of a crank drive rotating about the axis O, are balanced by a counterweight mounted on a separate flat spring. This spring, in addition to its other functions, serves as a guide by means of

which the load, for all practical purposes, can move only in a direction parallel to the direction of motion of the center of the cell on deflection. By means of a separate rod, the load is connected to its crank, fastened to the same axis O, as the crank of the cell, with the cranks of both cell and load having opposite directions relative to the axis O (see Fig.26).

In certain other instruments, the cell connected to the crank in the same fashion as in the altimeter, is balanced by a load attached directly to a lever fixed to the same axis O as the cell crank (see Fig.27).

It should be noted that the balancing of altimeter cells is more in accordance

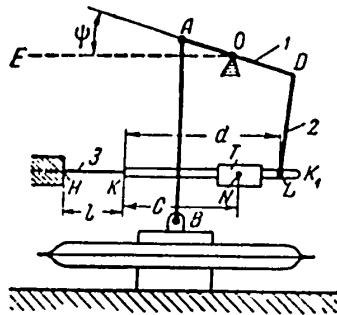


Fig.26 - Cell Balanced by the Load T,
Fastened to the Spring (3)

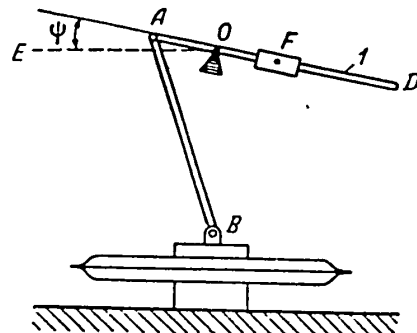


Fig.27 - Cell Balanced by the Load,
Fastened to the Crank Extension

with theory than balancing in accordance with the diagram in Fig.27. In actuality, only in case of accelerations acting parallel to the direction of movement of the load and cell, when the latter is deflected, will the two types of balancing function in approximately identical fashion, while for correct balance, the following equation has to be adhered to

$$Pc = Qb,$$

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where P is the reduced weight of the cell;

c is the length of the cell crank;

Q is the weight of the load;

b is the weight of the load lever.

The following case is also possible: When balanced as shown in Fig.27, the load lever and cell crank deviate from their center position, perpendicular to the direction of motion of the cell, by an angle ψ ; the acceleration j is perpendicular to the motion of the cell. In this case, the load Q creates a moment O , relative to the axis, this moment being equal to $\frac{j}{g} Qb \sin \psi = \frac{j}{g} Pc \sin \psi$. The cell either creates no moment whatever (if it is sufficiently rigid for deflection in the given direction) or a moment only a fraction of that discussed. It is quite obvious that this situation may substantially increase the instrument error due to inclination and acceleration.

In the balancing circuit of the pickup element shown in Fig.26, the counterweight spring (3), creating the determinate constant kinematic connection between the parts of this balancing system (i.e., in the hinges B, A, D, and L), must have adequate tension. It is essential that the force of this tension reduced to the pickup element is greater, at all instrument readings, than the product of the reduced force of unbalance of the pickup element (not yet balanced by the counterweight T) and the quantity $\frac{j_M}{g} + \cos \beta$, where j_M is the maximum acceleration possible under vibration, g is the acceleration of gravity, and β is the angle between the direction of deflection of the pickup element and the vertical, when the instrument is in the correct position.

This is due to the fact that the force of inertia is equal to the force of gravity, multiplied by the ratio $\frac{j}{g}$ (the latter being equally true of the reduced forces). However, in addition, here it is also necessary to consider the component of the force of gravity itself in the direction of motion of the pickup element, which is equal to this force multiplied by $\cos \beta$.

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In this case it is, of course, desirable that this adequate force of tension

of the counterweight spring is created primarily by an increased deflection of this spring. In other words, the rigidity of the spring is best as small as possible. This is necessary so that the force of tension at the spring undergoes less change as the crank rotates, i.e., as the instrument readings vary. Excessive tension on the spring may increase the errors of friction of the instrument. Moreover, the lower the rigidity of this spring, the lower will be the overall reduced total rigidity of all the elastic elements in the mechanism. Furthermore, the excessive increase in the rigidity of the elastic elements of instruments may, in a number of cases, prove undesirable.

It should be noted that, in certain diaphragm instruments (such as climb indicators), pressure cells of low rigidity are used as the pickup elements. In such instruments, the occasionally required rather high force of spring tension may significantly displace the average position of the diaphragm, the more so since the hair spring providing the constant kinematic connection between the other parts of the mechanism, has the function of providing a reduced moment about the crank axis in the same direction as the moment from the counterweight spring (3). This will be discussed below.

In addition, if we bear in mind the usual nonlinearity of the relationship between the deflection of the diaphragm and the pressure, such a displacement of the center position of the diaphragm may prove undesirable. With certain instruments, such as climb indicators, the diaphragms must be able to undergo symmetrical deflection in both directions, to yield either to reduced pressure on one side of the diaphragm or to increased pressure on the other side.

True, such displacement of the center position of the diaphragm can be prevented by means of an additional spring, acting directly on the diaphragm cell in ^{STAT} direction opposite to that of the counterweight spring (3) and to that of the hair spring. However, it must be borne in mind in this connection, that, in this arrangement, the reduced rigidity of the counterweight spring (3) of the hair spring,

and of the additional spring directly connected to the diaphragm cell, is added to the resistance of the cell itself, with the result that the system as a whole may prove to be undesirably resistant.

Here it should be noted that, in this latter case, the result is even less favorable for the type of balancing shown in Fig.27, despite the fact that the counterweight there has no spring. In actuality, with this type of balancing, the moment of the hair spring reduced to the axis of the crank must obviously equal the sum of the moments due to the action of the counterweight spring (3) and that of the hair spring when balancing is effected by the method shown in Fig.26. In this case, the additional stiffness of the hair spring required to obtain the supplementary moment replacing the moment of the counterweight spring (3) reduced to the diaphragm cell, must be considerably greater than the reduced rigidity of the counterweight spring (3), in the system of balancing shown in Fig.26.

This is due to the fact that the axis of the hair spring in instruments usually rotates several times as rapidly as the axis of the crank under examination. However, we know that when a force or moment is reduced, it is multiplied by the first power of the transmission ratio, and in any reduction of rigidity, the rigidity being reduced is multiplied by the square of the transmission ratio.

The latter may also be explained in another way: In view of the fact that the angle of rotation of a crank within the range of the instrument readings is significantly smaller than the angle of rotation of the part to which the hair spring is usually attached, it becomes a simple matter to select a counterweight spring (3) that will provide a significantly smaller ratio of deflections at maximum and minimum tension (within the operating range of the instrument) than the ratio of the angles of twist of the hair springs at maximum and minimum angles of tension. Since the supplementary reduced moment of the hair spring [replacing the effect of STAT the counterweight spring (3)] must be equated to the moment created by the counterweight spring at minimal tensions on the spring (3) or the hair spring, the follow-

ing may be derived:

- a) The rigidity of the counterweight spring reduced to the crank axis is smaller than the reduced additional rigidity of the hair spring;
- b) In view of the fact that the relative variation in tension of the counterweight spring is smaller than the relative variation in tension of the hair spring, the moment created by the counterweight spring at maximum tension will be less than the maximum additional moment of the hair spring (replacing the moment from the spring action), with the result that the total magnitude of the reduced force of friction of the entire mechanism of the instrument is somewhat lowered.

3. The moment of the hair spring, reduced to the crank axis, has to act in the same direction as the moment created by the counterweight spring (3); if this is not the case, these moments acting in opposite directions might completely cancel the tension needed on the shaft at the hinges B and A.

4. The design should allow for the possibility of varying the length b of the crank OD, to which the pin DL with the counterweight is attached, so as to correspond with variations in the length a of the crank OA of the transmission mechanism as the instrument is adjusted, with the object of approximately maintaining the ratio required after regulation.

This relationship takes the form

$$aP = b \frac{Q \left[l^2 + \frac{3}{2} l(c+d) + 3cd \right] + Q_1 \frac{l}{2} \left(-\frac{3}{4} l + d \right)}{l^2 + 3ld + 3dl^2}, \quad (4.37)$$

where P is the reduced weight of the pickup element (to the connection point of the shaft AB); Q is the weight of the counterpoise with the lever KK_1 ; l is the length of the free (working) part of the spring from the extreme point H of the fitting, to the point K at which the lever KK_1 attacks; c is the distance NK of the center of

gravity N of the counterweight and the lever KK_1 , from the point K at the end of the free portion of the spring; d is the distance of the point L of attachment of the pin LD from the same end, to the free portion of the spring; Q_1 is the weight of the free portion of a spring of length l . Equation (4.37) is written for a case in which the spring (3) is of constant cross section.

This equation may readily be derived from the following considerations. The force Q , applied to the center of gravity N of the counterweight, induces a deflection f_Q at point L equal to

$$f_Q = \frac{Q}{EJ} \int_0^l (l - c - x)(l + d - x) dx = \frac{Q}{EJ} \left[\frac{l^3}{3} + \frac{l^2}{2}(c + d) + lcd \right],$$

where J is the moment of inertia of the spring cross section;

E is the modulus of elasticity of the spring material.

The very weight of the spring (3) induces, at the point L , a deflection of the following value

$$f_{Q_1} = \frac{q}{EJ} \int_0^l \frac{(l - x)^2}{2} (l + d - x) dx,$$

where q is the weight of a unit spring length, i.e. $ql = Q_1$.

After integration we obtain

$$f_{Q_1} = \frac{q}{EJ} \frac{l^3}{2} \left(\frac{l}{4} + \frac{d}{3} \right) = \frac{Q_1}{EJ} \frac{l^2}{2} \left(\frac{l}{4} + \frac{d}{3} \right).$$

A force P_1 , applied at the point L , induces at that point a deflection equal to

$$f_{P_1} = \frac{P_1}{EJ} \int_0^l (l + d - x)^2 dx = \frac{P_1}{EJ} \left(\frac{l^3}{3} + l^2d + ld^2 \right) = \frac{P_1}{EJ} \frac{(l + d)^3 - d^3}{3}$$

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Balancing of the pickup element requires that

$$aP = bP_1,$$

where P_1 is obtained from the equation

$$f_Q + f_{Q_1} = f_{P_1},$$

and, therefore,

$$aP = bP_1 = b(f_Q + f_{Q_1}) \frac{P_1}{f_{P_1}},$$

from which we obtain eq.(4.37).

It should be mentioned that, in many instruments currently being manufactured, no provision has been made for changing the length of the lever OD in this fashion. Instead, many instruments are built to permit variations in the length d (by moving the attachment point L of the pin LD).

This makes it possible, of course, to balance the system but complicates the process of general regulation of the instrument, as compared to balancing, by changing the length of the lever OD. The point is that, in balancing by changing the length d, the angle between the pin and the direction of deflection of the pickup element (which is denoted by the letter ϵ) may be considerably larger than in balancing by means of changes in the length of the lever b. This follows from the consideration that, at a relative change in the length of the crank a (with no variation in the lever b), it becomes necessary to provide approximately the same relative variation in the $\frac{l + c}{l + d}$ ratio and, in view of the constancy of $l + c$, approximately the same relative variation in the dimension $l + d$. However, since the value of $l + d$ is comparatively large (by comparison with the quantities a and b), the absolute change in the magnitude of d may, in this case, be comparatively large. However, an increase in the ϵ angle has the following consequences:

1. At point D of the lever OD the force P_1 parallel to the pickup element will

be supplemented by a force $P_2 = P_1 \tan \epsilon$ perpendicular to the pickup, which will affect the instrument readings at positions of the crank at which the lever b is not perpendicular to the direction of the stroke of the pickup element.

In this case, the force in question, $P_2 = P_1 \tan \epsilon$ cannot be compensated when the cell is balanced, since the moment it creates is equal to $bP_2 \sin \alpha$, where α is the angle between the lever OD of the counterweight and the direction perpendicular to the deflection of the pickup element. In other words, the moment will differ at various angles α of rotation of the crank.

Furthermore, similar forces will develop in the case of accelerations, even when these accelerations are parallel to the stroke of the pickup element. They may attain a significant magnitude if the angle of rotation of the crank α is large enough. Of course, this also applies to vibrations.

Thus, even if the balancing of the pickup element by means of the counterweight T is done with much care, vibration may give rise to perturbing forces of significant magnitude.

From this it also follows that, in designs of this type, excessively large angles between the shafts and the direction of motion of the pickup element under deflection must not be permitted to arise. (This holds both for the crank OA and for the lever OD of the counterweight).

This is confirmed by the following considerations: In order for the pickup element to remain accurately balanced in all positions of the mechanism, it is necessary that the $\frac{dw}{dx}$ ratio of displacement w of the pickup element and the corresponding displacement x of the counterweight in a direction parallel to the displacement of the pickup element always remain approximately constant, regardless of the position of the mechanism. In a case in which the cranks OA and OD lie along a single straight line, the ratio of the rate of displacement of projections of tSTAT ends of the arms OA and OD of the lever in a single direction is, obviously, always a constant.

If, this being the case, the shafts of the given cranks have a direction of motion similar to the direction of the pickup element, the rates of displacement of the pickup element and the counterweight in the given direction will also be of an approximately constant ratio, since the cosines of small angles are very close to unity.

However, to render a lever, consisting of the cranks OA and OD, other than straight (as has sometimes been proposed) and then to determine, accordingly, what the position of the counterweight must be in order for the mechanism to be properly balanced in its various positions, is hardly rational in cases where the angles of rotation of the cranks are not excessively high, since such a calculation will be valid only for an average mechanism in a series. When each individual instrument is regulated, the constancy of the $\frac{dw}{dx}$ ratio will apparently be disrupted to a considerably greater degree than in regulation of a mechanism with a straight lever and shafts designed to be almost parallel to the motion of the pickup element.

2. When the ϵ angle is rather large, the tension on the counterweight spring (3) will differ from its tension at low ϵ angles; therefore, simultaneously with any variation in length d, the tension on this spring must be changed.

In view of this fact, balancing by changing the length of the lever OD is theoretically preferred.

It must be borne in mind, however, that when the instruments are adjusted and the length of the crank a is changed, a corresponding change must be made in the length of the lever OD (or in the length d, if such a method of balance is available). This can be confirmed, to a fair degree of approximation, in the following manner: The instrument is first placed in a position in which the direction of deflection of the pickup element is vertical. It is then rotated about its horizontal axis through 180° , in such a manner that the direction of deflection of the pickup element again becomes vertical, but in the opposite direction. Then, the instrument is tapped lightly, and a check is made as to whether the instrument

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readings, obtained from both directions, are identical.

This procedure is useful since the adjusters may be misled by the established engineering standards for instruments, since these provide only for the inclination of an instrument by a specific and rather small angle (30° , sometimes a little more).

5. All individual parts of the transmission mechanism from the crank to the axis of the hair spring must undergo careful and individual balancing.

6. It is desirable that the instrument mechanism provide some damping of its oscillations, if this does induce noticeable complications of the entire mechanism. An example of such damping in instruments with manometer cells is the use, indicated above, of a nozzle mounted in a stub pipe connecting the interior of the cells with the source of pressure being measured.

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PART TWO

CHOICE AND ANALYSIS OF VARIOUS MECHANICAL COMPONENTS OF INSTRUMENTS

CHAPTER V

SPRING COMPONENTS OF CONSTANT RIGIDITY

Section 1. Bourdon Tubes

As a comment, valid for Chapter Five as a whole, it may be noted that a truly constant linear ratio of load to deflection (spring) is a general property of the simplest spring elements of constant rigidity only when the deflection is infinitesimal and the load equally small. The equations for calculating the majority of springs are essentially based on the case of infinitesimal deflection, but they are applied, in approximated form, to deflections of measurable size in view of the fact that, for deflections within given limits, these spring components undergo insignificant changes in configuration and rigidity.

In accordance with the formulas derived by V.I.Feodos'yev* for thin-walled Bourdon tubes, we have the following relation, when a difference p exists between the pressures within and outside the tube:

$$\left(\frac{\Delta(d\theta)}{d\theta} \right)_p = p \frac{1-\mu^2}{E} \frac{t^2}{bh} \left(1 - \frac{b^2}{a^2} \right) \frac{a}{\beta + \lambda^2}, \quad (5.1)$$

where $d\theta$ is the angle between two cross sections of the tube at infinitesimal spacing;

$\Delta(d\theta)$ is the variation in the angle under the effect of the differential pres- STAT

*These questions are dealt with in greater detail in V.I.Feodos'yev's Spring Components of Precision Instruments, Oborongiz (Defense Press), 1949.

sure p ;

E is the modulus of elasticity of the tube material;

μ is Poisson's modulus;

ρ is the radius of curvature by which the tube is bent;

a and b are the lengths of the radii of the tube cross section;

h is the thickness of the tube walls;

α and β are dimensionless factors, whose magnitudes are functions of the $\frac{a}{b}$ ratio.

These are given in Tables 1 and 2, respectively, for cases of tubes of elliptical and oval cross section,

$$\lambda = \frac{\rho h}{a^2}.$$

On the basis of eq.(5.1) it is possible to determine the magnitude and direction of the corresponding displacement of the free end of the tube.

If the free end of a Bourdon tube, bent into an arc of a circle ($\rho = \text{const}$), is taken as the origin of coordinates, and if the Ox and Oy axes of the coordinates are represented by any mutually perpendicular straight lines passing through this point in the plane of symmetry of the tube (see Fig.2), the displacements f_x and f_y in the directions Ox and Oy will be

$$\left. \begin{aligned} (f_x)_p &= - \int y \Delta(d\theta) = -p \frac{1-\mu^2}{E} \frac{\rho^2}{bh} \left(1 - \frac{b^2}{a^2}\right) \frac{a}{\beta + \lambda^2} \int y d\theta; \\ (f_y)_p &= \int x \Delta(d\theta) = p \frac{1-\mu^2}{E} \frac{\rho^2}{bh} \left(1 - \frac{b^2}{a^2}\right) \frac{a}{\beta + \lambda^2} \int x d\theta, \end{aligned} \right\} \quad (5.2)$$

where x and y are coordinates of the instantaneous point of intersection of the longer axis (a) of the corresponding cross section of the tube, and its plane of symmetry.

Here the integral is taken within limits for the entire length of the tube. It is entirely obvious that, given a constant radius of curvature ρ , the magnitudes of

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the x and y coordinates at any point in the given periphery can be expressed as functions of the angle θ , between the tangent at this point and some given constant

line of action (see Section 3, Chapter I). This yields expressions capable of simple integration.

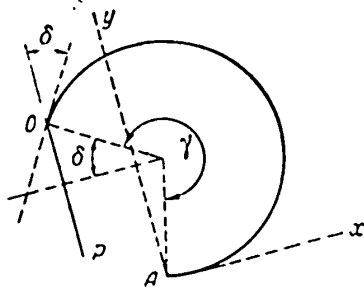


Fig.28 - System of Coordinates Used in Determining the Degree and Direction of Deflection of Bourdon Tubes

If we take, as the origin of coordinates, the point A at which the tube is mounted to its base, and as the axis of coordinates any two mutually perpendicular straight lines passing through this point (Fig.28), eq.(5.2) will take on the form

$$\left. \begin{aligned} (f_x)_p &= -p \frac{1-\mu^2}{E} \frac{\rho^2}{bh} \left(1 - \frac{b^2}{a^2} \right) \frac{a}{\beta + \lambda^2} \int (y_0 - y) d\theta \\ (f_y)_p &= p \frac{1-\mu^2}{E} \frac{\rho^2}{bh} \left(1 - \frac{b^2}{a^2} \right) \frac{a}{\beta + \lambda^2} \int (x_0 - x) d\theta \end{aligned} \right\} \quad (5.3)$$

Here x_0 and y_0 are the coordinates of the free end of the tube in its initial position.

For cases in which the Bourdon tube is deflected by the action of an external moment M , the same work by V.I.Feodos'yev yields

$$\left[\frac{\Delta(d\theta)}{d\theta} \right]_M = M \frac{1-\mu^2}{E} \frac{\rho}{k_e J_a}, \quad (5.4)$$

where J_a is the moment of inertia of a tube cross section relative to its diameter, $2a$;

k_e is a correction allowing for the phenomenon noted by Karman for hollow tubes on deflection (and described in courses on the theory of elasticity).

Table 1
Data for Calculating Bourdon Tubes of Elliptical Cross Section

$\frac{a}{b}$	1	1,5	2	3	4	5	6	7	8	9	10	∞
α	0,750	0,636	0,566	0,493	0,452	0,430	0,416	0,406	0,400	0,395	0,390	0,368
β	0,083	0,062	0,053	0,045	0,044	0,043	0,042	0,042	0,042	0,042	0,042	0,042
ζ	0,75	0,602	0,530	0,454	0,415	0,392	0,382	0,377	0,372	0,367	0,362	0,338
ξ	0,833	0,662	0,584	0,499	0,459	0,439	0,429	0,423	0,416	0,410	0,404	0,381
n	0,197	0,149	0,142	0,121	0,111	0,106	0,102	0,100	0,098	0,097	0,095	0,089
$J_0 - J_1$	0,785	0,702	0,705	0,680	0,675	0,671	0,670	0,670	0,670	0,670	0,669	0,667

Table 2
Data for Calculation of Bourdon Tubes of Oval Cross Section

$\frac{a}{b}$	1	1,5	2	3	4	5	6	7	8	9	10	∞
α	0,637	0,594	0,548	0,480	0,437	0,408	0,388	0,372	0,360	0,350	0,343	0,267
β	0,096	0,110	0,115	0,121	0,121	0,121	0,121	0,120	0,119	0,119	0,118	0,114
ζ	0,713	0,604	0,539	0,470	0,430	0,403	0,388	0,369	0,356	0,347	0,341	0,267
ξ	0,811	0,713	0,652	0,591	0,552	0,524	0,504	0,488	0,476	0,467	0,459	0,296
n	0,159	0,151	0,144	0,131	0,122	0,115	0,110	0,107	0,105	0,103	0,101	0,083

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In accordance with V.I. Feodos'yev's derivations, this correction, for tubes of elliptical and oval cross section, is equal to

$$k_s = 1 - \frac{\zeta}{\xi + \lambda^2}.$$

The magnitude of the dimensionless factors ζ and ξ , constituting functions of the $\frac{a}{b}$ ratio are presented in Tables 1 and 2 for tubes of elliptical and oval cross section, respectively.

For elliptical sections, the moment of inertia J_a is

$$J_a = 4ab^2h(J_0 - J_2),$$

and Table 1 derives the values of $J_0 - J_2$ for various $\frac{b}{a}$ ratios.

Where an oval cross section is concerned, the moment of inertia J_a equals

$$J_a = 4b^3h \left(\frac{a}{b} - 1 + \frac{\pi}{4} \right).$$

On the basis of eq.(5.4) it is possible to determine the magnitude and direction of displacement of the free end of the Bourdon tube under the action of a concentrated force P , applied in any direction to the free end of the tube. If the origin of coordinates is set at the free end of the tube, the Oy axis in the direction of the force P and the Ox axis in the perpendicular direction (see Fig.2), the displacements $(f_x)_p$ and $(f_y)_p$ will be

$$\left. \begin{aligned} (f_x)_p &= -P \frac{1-\mu^2}{E} \frac{P}{k_s J_a} \int xy d\theta; \\ (f_y)_p &= P \frac{1-\mu^2}{E} \frac{P}{k_s J_a} \int x^2 d\theta. \end{aligned} \right\} \quad (5.5)$$

However, if the origin of coordinates is taken as the point of attachment of the tube, and the direction of the axes is the same as in eq.(5.3), we obtain

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$$\left. \begin{aligned} (f_x)_P &= -P \frac{1-\mu^2}{E} \frac{\rho}{k_e J_a} \int (x_0 - x) (y_0 - y) d\theta; \\ (f_y)_P &= P \frac{1-\mu^2}{E} \frac{\rho}{k_e J_a} \int (x_0 - x)^2 d\theta. \end{aligned} \right\} \quad (5.6)$$

The limits of integration in eqs.(5.5) and (5.6) are, of course, the full length of the tube.

Note: Equations (5.5) and (5.6) are readily derived on the basis of eq.(5.4), and eqs.(5.2) and (5.3) on the basis of eq.(5.1), by methods generally known and set forth in courses on the theory of elasticity (one variant of derivations of this type was derived in the Section on allowance for errors of friction). For this reason, these derivations are not made here.

Calculation of particular instruments may sometimes require knowledge of the magnitude of the variation ΔV in the internal volume of the tube on straightening under the effect of differential pressure. In accordance with the V.I.Feodos'yev derivation, this variation in volume equals

$$\Delta V = 12p \frac{1-\mu^2}{E} \frac{\rho^3}{h} a \gamma \left(1 - \frac{b^2}{a^2} \right) \frac{n}{\beta + \lambda^2}.$$

Here γ is the overall angle of the arc in which the tube is bent. The value of the dimensionless magnitude n , dependent on the $\frac{a}{b}$ ratio, is given in Tables 1 and 2.

In order to determine the stresses in the substance of a Bourdon tube of elliptical cross section on deformation under the effect of differential pressure p , the following equations are derived in V.I.Feodos'yev's work on the subject:

$$\left. \begin{aligned} \sigma_1 &= p \frac{\rho}{h} \left(1 - \frac{b^2}{a^2} \right) \frac{1}{\beta + \lambda^2} \left[(6\omega - \alpha \cos \varphi) \pm 3\mu \lambda \left(\frac{J_2}{J_0} - \sin^2 \varphi \right) \right]; \\ \sigma_2 &= p \frac{\rho}{h} \left(1 - \frac{b^2}{a^2} \right) \frac{1}{\beta + \lambda^2} \left[3\lambda \left(\frac{J_2}{J_0} - \sin^2 \varphi \right) \pm \mu (6\omega - \alpha \cos \varphi) \right], \end{aligned} \right\} \quad (5.7) \quad \text{STAT}$$

where σ_1 is the stress in the meridional line of the tube, i.e., in planes passing through the axis of curvature around which the tube is bent in a circular arc;

σ_2 is the stress in perpendicular direction, i.e., in directions parallel to the plane of symmetry of the tube;

φ is the angle by which the coordinates z and y of any point in the elliptical cross section of the tube (measuring from the middle of the tube wall thickness) may be expressed in the form given in Fig.29

$$\begin{aligned} z &= a \sin \varphi; \\ y &= b \cos \varphi. \end{aligned}$$

The quantities J_2 and J_0 in eqs.(5.7) are elliptical integrals of the form

$$\begin{aligned} J_0 &= \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \varphi} d\varphi; \\ J_2 &= \int_0^{\frac{\pi}{2}} \sin^2 \varphi \sqrt{1 - k^2 \sin^2 \varphi} d\varphi, \end{aligned}$$

where $k^2 = 1 - \frac{b^2}{a^2}$. They are, therefore, constants for the given k of each specific tube.

Expression ω in eq.(5.7) is a highly complex function of the angle φ and the tube parameters. Table 3 adduces the values for this function, according to V.I. Feodos'yev.

In this Table, ψ is a supplementary parameter of the following value:

$$\psi = \arcsin k.$$

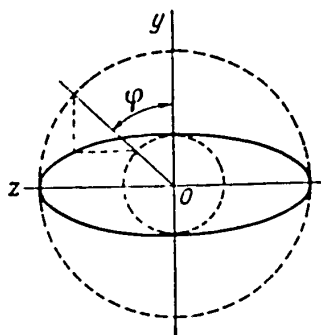
Thus, each given value of k , i.e., each individual tube, corresponds to a specific value of ψ and thus to a particular column of values of the magnitude ω ; however, these ω values vary with various values of the angle φ . Since these values

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are given in the Table rather than by an analytical expression, the usual method of finding the maximum value of expression (5.7) by the φ angle is not applicable. Therefore, it is merely necessary to calculate the magnitude of the stresses σ_1 and σ_2 for various values of the angle φ , and from this to determine the approximate magnitude of the maximum stresses occurring.

For thick-walled Bourdon tubes of greatly elongated cross section, V.I. Feodos'yev provides, in his study, the following

formula for straightening under the effect of the differential pressure p :



$$\left[\frac{\Delta(d\theta)}{d\theta} \right]_p = p \frac{1-\mu^2}{E} \frac{a^2}{bh} \frac{1-\gamma}{\frac{h^2}{12b^2} - \chi}, \quad (5.8)$$

where

$$\chi = \frac{1}{ca} \frac{\operatorname{sh}^2(ca) \cdot \sin^2(ca)}{\operatorname{ch}(ca) \operatorname{sh}(ca) \cos(ca) \sin(ca)},$$

Fig.29 - Relationship between the z and y Coordinates of an Ellipse and the Angle φ

while

$$ca = \sqrt[4]{\frac{3a^4}{\rho^2 h^2}} = \sqrt[4]{\frac{3}{\lambda^2}}.$$

For a deflection due to an external moment, we have the equation

$$\left[\frac{\Delta(d\theta)}{d\theta} \right]_M = \frac{M}{4} \frac{1-\mu^2}{E} \frac{\rho}{ab^2 h} \frac{1}{\frac{h^2}{12b^2} - \chi}. \quad (5.9)$$

Section 2. Corrugated Tubes (Sylphon)

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In V.I. Feodos'yev's study Spring Components in Precision Instrument Manufac-

the following equation is given for determining the deflection f of

Table 3
Magnitudes of ω

ψ	30	50	60	65	70	75	80	85
0	0,15631	0,13751	0,12437	0,11694	0,10909	0,10142	0,09328	0,08672
5	0,15448	0,13581	0,12278	0,11541	0,10762	0,10002	0,09194	0,08543
10	0,14911	0,13084	0,11811	0,11091	0,10331	0,09589	0,08802	0,08167
20	0,12906	0,11243	0,10089	0,09438	0,08750	0,08081	0,07369	0,06795
30	0,10045	0,08648	0,07680	0,07142	0,06569	0,06013	0,05417	0,04937
40	0,06903	0,05853	0,05129	0,04723	0,04293	0,03877	0,03425	0,03061
50	0,04047	0,03311	0,02905	0,02642	0,02363	0,02094	0,01794	0,01552
60	0,01891	0,01547	0,01308	0,01173	0,01027	0,00892	0,00726	0,00594
70	0,00600	0,00484	0,00402	0,00355	0,00304	0,00257	0,00194	0,00144
90	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000

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a sylphon under the effect of the differential pressure p :

$$f = p S_{\text{eff}} \frac{1 - \mu^2}{E h_0} \frac{n}{A_0 - \alpha A_1 + \alpha^2 A_2 + B_0 \frac{h_0^2}{R_0^2}} \quad (5.10)$$

Here n is the number of complete leaves;

S_{eff} is the effective area of the sylphon, taken here to be

$$S_{\text{eff}} = \pi \left(\frac{R_0 + R_i}{2} \right)^2,$$

where R_0 is the outside radius of the sylphon,

R_i is the inside radius (Fig.30);

$$\alpha = \frac{4 r n_1 - 2r - L}{2 (n_1 - 1) (R_0 - R_i - 2r)},$$

where n_1 is the number of external truncations of the leaves (i.e., in Fig.30 $n_1 = 6$);

r is the radius of curvature at the points at which the material is corrugated into a leaf bellows (see Fig.30);

L is the length of the sylphon from the top edge of the upper outside curvature to the bottom edge of the lower outside curvature (see Figs.30 and 33).

The values of the factors B_0 , A_0 , A_1 , and A_2 are given in Figs.31 and 32.

The classification of the spring components of linear characteristics also includes certain diaphragms and diaphragm chambers for pressure measurement (when the appropriate form of corrugation is used, rather deep corrugation usually being required). In view of the fact that it is only particular types of diaphragms that possess such linear characteristics, while the great majority present nonlinear characteristics (stiffness increases with deflection), we will present all points pertaining to diaphragms in the Chapter on spring components of variable stiffness. STAT

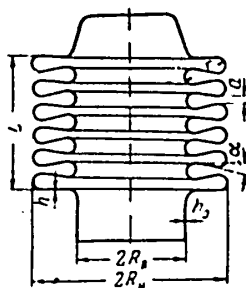


Fig.30 - Axial Cross Section of Sylphon

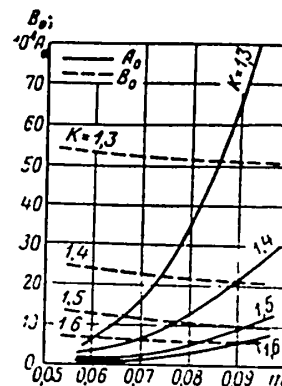


Fig.31 - Ratio of the Factors A_0 and B_0 to the values of k and m in the Equation for Calculating the Deflection of Sylphons

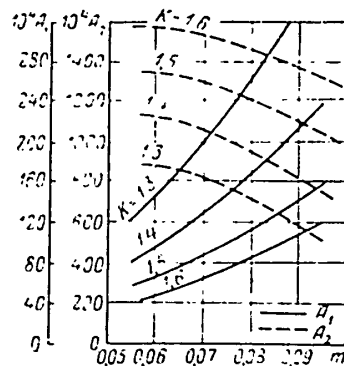


Fig.32 - Ratio of Factors A_1 and A_2 to the Values of k and M in the Equation for Calculating the Deflection of the Sylphons

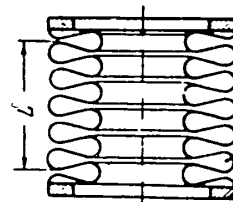


Fig.33 - Method of Determining the Length L in the Equation for Calculating Sylphons, When the Outer Leaves are Incomplete

Section 3. Coil-Action Springs (Remarks on Calculations)

Equations for calculating cylindrical coil-action helical springs, in which the spring as a whole expands or contracts, have the following form:

$$\left. \begin{aligned} f &= P \frac{64nR^3}{Gd^4}; \\ \tau &= \frac{16PR}{\pi d^3}, \end{aligned} \right\} \quad (5.11)$$

where P is the force expanding or compressing the spring;

f is the corresponding deflection of the spring;

τ is the maximum shear stress in the spring material permissible at this point;

R is the radius of the turns of the spring (measuring from the center of the spring wire);

n is the number of turns;

G is the shear modulus of the spring material;

d is the diameter of the wire.

Here we derive only equations for springs made of wire of circular cross section, since this type is virtually the only type used in the manufacture of instruments.

However, it may occasionally happen, in calculations for such springs, that particular questions arise, which cannot always be answered with adequate clarity from the general method of derivation of the formulas.

By way of example, let us examine one such question: This question is of a certain significance in principle since it sheds some light on the physical essence of the action of the type of springs under discussion.

We know that such springs function accurately without skewing, and that the general formulas for calculation are applicable, if the applied load acts along ^{STAT} the central axis of the spring. Therefore, this load must not be applied directly to the end of the last turn of the spring, since this will cause the spring to skew.

In view of the foregoing, a bend is usually made from the end of the final turn of the spring (particularly when the operation of the spring results in its expansion) to the center of the spring coils. This extension is very often obtained simply by straightening the end of the wire of which the spring has been made.

In calculating such springs, allowance is sometimes made not only for the deflection of the helical portion of the spring but also for these extensions at its ends. True, this deflection is usually very small relative to the deflection of the helical portion of the spring and is often ignored, but sometimes it is essential that it be taken into account.

In this connection, the following question was raised: Is it possible, in calculating the deflection of the extensions to the middle of the spring, to regard them as straight or curved beams fixed at one end? The question here is based on the consideration that the spring operates by coil action, i.e., the cross sections of the helical portion of the spring rotate in some fashion relative to each other, and it would therefore appear that these extensions to the center of the spring cannot be regarded as beams with fixed ends, since rotation of the point of attachment must also be allowed for.

However, as demonstrated below, there is no foundation for this doubt. Let us first examine this question as it applies to cylindrical springs.

At any cross section of the helical portion of such a spring, the diameter normal to the axial line of the cylinder forming the spring naturally remains perpendicular to this axial line after deflection of the spring. If this were not true, either the spring as a whole, or portions thereof, would skew. This is impossible, of course, because of the physical law governing the phenomenon in question. In actuality, as a spring is deflected, each cross section rotates about a given diameter of this cross section, perpendicular to the axial line of the spring. This rotation is proportional, by an angle, to the change in the inclination of the helical line of the spring upon deflection.

In addition, small rotations of these cross sections of the spring about axes parallel to the central axis of the spring may also occur, in which the given diameters of these cross sections are not taken out of planes normal to the central axis of the spring.

Thus, these diameters of the cross sections of the spring do not change in angle relative to the central axis of the spring, while the cross sections rotate relative to each other as the spring is deflected, due to the fact that the spring operates by coil action. It is not difficult to find an explanation for the co-existence of these two apparently contradictory propositions.

Let us take two cross sections of a spring in which the diameters normal to the central axis of the spring constitute an angle of infinitesimal size $d\varphi$. Thus, the length dl of the spring segment between these two sections is

$$dl = \frac{R d\varphi}{\cos \alpha}, \quad (5.12)$$

where R is the radius of the spring turns;

α is the angle of inclination of the helical line of the spring.

Figure 34 shows projections, onto a plane perpendicular to the central axis of the spring, of two diameters AB and CD , also normal to the central axis O of the spring, at the two cross sections of the spring, separated by an infinitesimal distance.

Let us start by examining the case of an infinitesimal increase in spring deflection, corresponding to an infinitesimal increase in load dP . When this is the case, the angle α of the helical line of the spring undergoes an infinitesimal change $d\alpha$.

In accordance with the conditions adopted, let the two sections of the spring under examination rotate through an angle $d\alpha$ as the spring is deflected, about $STAT$ two diameters normal to the axis of the spring. However, these diameters AB and CD are not parallel. Therefore, let us divide the rotations of the cross sections

about these diameters into several turns.

In view of the small size of the turns in question (it having been stated that the angle $d\alpha$ is infinitesimal), they may be laid out along the "component" turns in

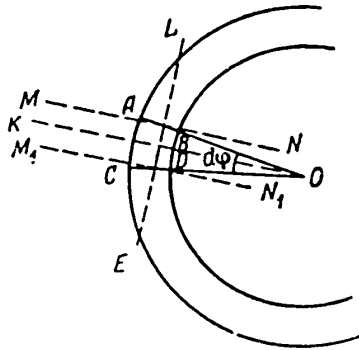


Fig.34 - Data for Investigation of the Nature of Deformation of a Cylindrical Helical Coil-Action Spring

the same manner as the velocity of the turns (this would be impossible, of course, if the turns were large). Thus, let us at the outset, divide the turns of the sections of the spring around the straight lines AB and CD into turns $d\gamma_1$ around the straight lines MN and M_1N_1 , parallel to the bisectrix of the angle $d\phi$, and turns $d\gamma_1$ around the straight line EL, normal to the straight lines MN and M_1N_1 .

As indicated, the small size of the angles of rotation makes it possible to write

$$d\alpha_1 = d\alpha \cos \frac{d\varphi}{2} \approx d\alpha;$$

$$d\gamma_1 = d\alpha \sin \frac{d\varphi}{2} \approx \frac{d\alpha d\varphi}{2}.$$

It should be noted that rotations through an angle $d\alpha_1$ about the axes MN and M_1N_1 have the same direction for both cross sections. Rotations through an angle $d\gamma_1$ about the axis EL have opposite positions, at the two cross sections under examination. It is these two latter turns that induce some degree of skewing in the case of the spring component under examination.

In order to determine the magnitude of the angle of twist at two cross sections, separated by a distance of dl , the rotation $d\gamma_1$ must further be resolved into the following components:

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a) Rotation about the straight line coinciding with the direction of the spring segment under examination or, in other words, forming an angle α with the plane of the paper in Fig.34;

b) Rotation about a straight line normal to the plane of the paper.

When the foregoing resolution has been performed, the rotation $d\gamma$ about a straight line, coinciding with the direction of this spring segment, is equal to

$$d\gamma = \frac{d\gamma_1}{\cos \alpha} = \frac{dz dz}{\cos \alpha}$$

Thus, the two sections under examination show a twist, relative to each other, by an angle $d\mu$, equaling

$$d\mu = 2d\gamma = \frac{2dz dz}{\cos \alpha} \quad (5.13)$$

We will demonstrate below, as a result of the assumptions made, i.e., obeying eq.(5.13), that we obtain the same formula for total deflection df of the spring, under the effect of the force dF , as when the magnitude of this deflection is arrived at in the usual way.

The moment of twist dM acting on the cross section of the spring, is equal to

$$dM = dPR \cos \alpha.$$

Under the effect of this moment, the element dl of the spring is twisted by an angle $d\mu$, equal to

$$d\mu = \frac{dM dl}{GJ_0} = \frac{dPR \cos \alpha dl}{GJ_0},$$

where J_0 is the moment of inertia of the spring cross section relative to the center of this section;

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G is the shear modulus of the spring material. We know that

$$J_0 = \frac{\pi d^4}{32},$$

where d is the diameter of the wire of which the spring is made.

Therefore, substituting the expression dl from eq.(5.12) ,

$$d\mu = \frac{32 R \cos \alpha dl}{G\pi d^4} dP = \frac{32 R^2 d\varphi}{G\pi d^4} dP. \quad (5.14)$$

Comparing eqs.(5.13) and (5.14), we obtain

$$\frac{d\varphi d\alpha}{\cos \alpha} = \frac{32 R^2 d\varphi}{G\pi d^4} dP,$$

from which we derive

$$d\alpha = \frac{32 R^2 \cos \alpha}{G\pi d^4} dP. \quad (5.15)$$

The height dh of the elementary segment dl of the helical portion of the spring, i.e., the projection of this segment onto the central axis of the spring, comes to

$$dh = dl \sin \alpha.$$

When the angle α varies by a magnitude $d\alpha$, we obtain

$$d(dh) = dl \cos \alpha d\alpha = R d\varphi d\alpha = dP \frac{32 R^3}{G\pi d^4} \cos \alpha d\varphi.$$

However, the overall change dh_c in the deflection of the entire spring, when the load on dP is varied, comes to

$$dh_c = dP \frac{32 R^3}{G\pi d^4} \int_0^{\varphi_c} \cos \alpha d\varphi, \quad (5.16) \quad \text{STAT}$$

where φ_c is the total angle of the coil of the spring.

If the spacing of the spring coils is constant ($\alpha = \text{const}$), we have

$$dh_c = dP \frac{64 n R^3}{G d^4} \cos \alpha, \quad (5.17)$$

where n is the coil of the spring.

Thus we obtained the same formula for the flexure dh_c of the spring that is obtained by the usual derivation of the value of this flexure due to the torsional moment $dP \cdot R \cos \alpha$, acting on the spring. As is generally known, in deriving the formula for the flexure of such a spring under the action of the load P , using the method ordinarily employed, we get the same result, i.e., that the flexure f_1 of the spring due to the torsional moment so produced, is equal to

$$f_1 = P \frac{64 n R^3}{G d^4} \cos \alpha$$

(this equation corresponds to the case when the value of the angle α is the same over the entire length of the spring). However, in addition, we now get a certain additional flexure f_2 of the spring, due to the flexure of its elements, which is equal approximately to:

$$f_2 = P \frac{128 n R^3}{E d^4} \frac{\sin^2 \alpha}{\cos \alpha}, \quad (5.18)$$

where E is the modulus of elasticity of the spring material.

Thus, in accordance with the given derivation, the total deflection is approximately

$$f = P \frac{64 n R^3}{G d^4} \left(\cos \alpha + \frac{2G}{E} \frac{\sin^2 \alpha}{\cos \alpha} \right). \quad (5.19)$$

Analogously, by using the same derivations for the case in which the angle α is not constant along the total length of the spring, and is some function of the angle φ , we obtain

$$f = P \frac{32 R^3}{G \pi d^4} \int_0^{\varphi_c} \left(\cos \alpha + \frac{2G}{E} \frac{\sin^2 \alpha}{\cos \alpha} \right) d\varphi, \quad (5.20) \text{ STAT}$$

where φ is the angle of coil of a portion of the spring from its end to the given

component $d\varphi$;

φ_c is the total angle of coil of the entire spring.

However, to calculate these springs it has become the custom to make approximate use of eqs.(5.11), in the first place since the angle α of these springs is usually not large, and therefore $\cos \alpha$ differs little from unity, and in the second place since the presence of the second term in parentheses in eqs.(5.19) and (5.20) compensates, to some degree, for this difference between unity and the magnitude of $\cos \alpha$.

Moreover, this derivation ignores the spring deformation due to shearing stresses in a direction normal to the cross section of the wire and due to tensile stress along the center line of the wire, something that is entirely permissible in view of the insignificance of these deformations.

It should further be noted that the derivation of eqs.(5.19) and (5.20) is performed only for infinitesimal loads dP and deflections df . True, the deflection f of a spring, resulting from the action of a finite force P may be presented as the sum of the infinitely small deflections df . However, this does not eliminate the fact that there is some incongruity inherent in the substitution, in eqs.(5.11), of magnitudes f and P for df and dP for the reason that, in gradual deflections of a spring, slight changes in the magnitudes of the angle α , radius R , and the number of turns in the spring are always taking place. Therefore, this equation, like the equations for the majority of other types of springs, can be used only for approximations, where spring deflections of minor magnitude are involved.

The same derivations yield the following formulas for calculating noncylindrical helical coil-action springs:

$$\left. \begin{aligned} f &= P \frac{32 \int_0^{\varphi_c} R^3 d\varphi}{G\pi d^4}; \\ \tau &= \frac{16 PR_{\max}}{\pi d^3}, \end{aligned} \right\} \quad (5.21) \quad \text{STAT}$$

where the variable radius R of the spring turns is expressed as some specific function of the angle φ (depending on the configuration of the spring) and where $R_{\max}$ is the maximum radius of the spring.

The derivation of these last formulas allows for all the approximations used in deriving eq.(5.11).

Note: The formulas determining the maximum tension in a spring, both in eq.(5.11) and in eq.(5.21) are written for cases of central loading of the spring along its central axis.

Let us examine the degree to which the above remark, as to the nature of the action of cylindrical springs and their extensions to the axial line thereof, are also valid for noncylindrical springs.

The action of these springs, as observed in practice, also gives reason to assume that we are dealing here with a phenomenon analogous to that described above for that of cylindrical springs. That is to say, it would appear that we have the following situation: If we imagine the cross section of any point of any turn of a spring as a plane passing through the central axial line of the entire spring, along which the load on the spring is acting, and if at this cross section we note the diameter normal to the axial line of the spring (the cross section having the appearance of an ellipse only slightly from a circle), then, after the spring has been deflected, this spring diameter will remain perpendicular to the central axis of the spring, and the spring cross section in question will rotate about this diameter at some angle.

In addition, here, as with cylindrical springs, small rotations of the sections of the spring may occur about axes parallel to the central axis of the spring, these cross-sectional diameters not being emergent from planes normal to the central axis of the spring.

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Let us designate by $d\alpha$ the angle of rotation of the section about the diameter defined above, when the load on the spring varies by dP .

It is obvious that, in the given case (i.e., for noncylindrical springs), the angle $d\alpha$ is not identical for various sections of the spring and constitutes some function of the position of the section under consideration, relative to the axial line of the spring. The angle $d\alpha$ is further related as follows to changes in the configuration of the spring under deflection.

If we denote by β the angle which the given spring component (measured longitudinally along the wire of which the spring is made) makes with the plane normal to the axial line of the spring, and if $d\beta$ is used to designate the change in this angle with any change in the load on the spring, then

$$d\alpha \approx \frac{d\beta}{\cos \delta},$$

where $\frac{\pi}{2} - \delta$ is the angle between the plane of the cross section (i.e., the plane passing through the axial line of the spring) and the projection of the turn of the spring at the given point onto a plane normal to the axial line of the spring. In the majority of springs, the angle δ is small, and therefore $\cos \delta$ is very near unity. Identically,

$$\operatorname{tg} \alpha = \frac{\operatorname{tg} \beta}{\cos \delta} = \frac{dh}{R d\varphi}, \quad (5.22)$$

where R is the distance of the center of the spring section in question from the axial line of the spring;

dh is the magnitude of the rise of the spring in the segment corresponding to the angle $d\varphi$ (i.e., the projection of that component of the spring onto the axial line);

$d\varphi$ is the angle between two planes passing through the axial line of the spring and bounding the spring component in question ($d\varphi$ representing the same magnitude as in the case of cylindrical springs).

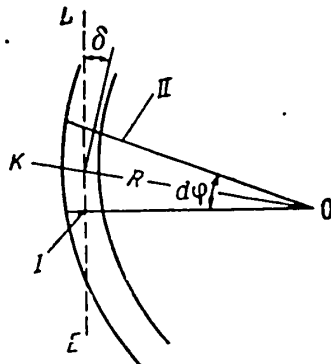
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Obviously, the angle α may also differ for various spring components.

Here, as in the case of the cylindrical spring, let us lay out rotations about the diameters of two adjacent sections of the spring (considering these rotations to be small) for rotations in a direction parallel to the bisectrix OK of the angle constituting projections of these diameters onto a plane normal to the axial line of the spring, and for rotations in a direction parallel to the projection EL of the given spring element onto the same plane normal to the axial line of the spring.

In the second of these directions, we will have (Fig.35) the following rotations:

a) In the cross section I there will be a rotation $d\gamma_1$, determined from the equation



$$\frac{d\gamma_1}{\sin \frac{d\varphi}{2}} = \frac{d\alpha_1}{\sin \left(\frac{\pi}{2} - \delta \right)};$$

$$d\gamma_1 = \frac{d\alpha_1 \sin \frac{d\varphi}{2}}{\cos \delta} \approx \frac{d\alpha_1 d\varphi}{2 \cos \delta},$$

where $d\alpha_1$ is the rotation about an axis parallel to the axis OK;

$d\gamma_1$ is the rotation about the axis EL;

Fig.35 - Data on the Nature of Deformations of Noncylindrical Helical Coil-Action Springs

b) In the cross section II there will be a rotation $d\gamma_2$ about the axis EL in the opposite direction, equal to

$$d\gamma_2 = -\frac{d\alpha_2 \sin \frac{d\varphi}{2}}{\cos \delta} \approx -\frac{d\alpha_2 d\varphi}{2 \cos \delta},$$

where $d\alpha_2$ is the rotation about the axis parallel to the OK axis.

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Plotting these values on rotations in the direction of the given spring component, and on rotations in the direction parallel to the axial line of the spring as

a whole, we obtain; for a cross section I, a rotation $d\gamma_{11}$ in the direction of the spring component, having the value:

$$d\gamma_{1,1} = \frac{d\gamma_1}{\cos \beta} = \frac{d\alpha_1 d\varphi}{2 \cos \beta \cos \delta},$$

and, for a cross section II, a rotation $d\gamma_{2,1}$ of the magnitude

$$d\gamma_{2,1} = \frac{d\gamma_2}{\cos \beta} = -\frac{d\alpha_2 d\varphi}{2 \cos \beta \cos \delta}.$$

Thus we obtain a twist of the component by an angle $d\theta$, equal to

$$d\mu = d\gamma_{1,1} - d\gamma_{2,1} = \frac{(d\alpha_1 + d\alpha_2) d\varphi}{2 \cos \beta \cos \delta}.$$

When the angle $d\varphi$ is infinitesimal and small quantities of higher powers are neglected, we may write

$$d\mu = \frac{d\alpha d\varphi}{\cos \beta \cos \delta}. \quad (5.23)$$

The variation dM , induced in the moment of twist for the spring element, when the load is varied by dP , is equivalent to

$$dM = dP R \cos \beta \cos \delta.$$

The length dl of the spring element will become

$$dl = \frac{R d\varphi}{\cos \beta \cos \delta}.$$

Therefore, the angle of twist $d\mu$ equals

$$d\mu = \frac{dM dl}{GJ_0} = \frac{dP R^2 d\varphi}{GJ_0} = \frac{32 R^2 d\varphi}{G\pi d^4} dP. \quad (5.24) \text{ STAT}$$

A comparison of eqs. (5.23) and (5.24) yields

$$dx = \frac{32 R^3 \cos \beta \cos \delta}{G \pi d^4} dP. \quad (5.25)$$

Further,

$$dh = dl \sin \beta.$$

When the angle β is varied by $d\beta$, we have

$$\begin{aligned} d(dh) &= dl \cos \beta d\beta = \frac{R d\varphi}{\cos \delta} d\beta = \frac{R d\varphi}{\cos \delta} d\alpha \cos \delta = \\ &= \frac{32 R^3 \cos \beta \cos \delta d\varphi}{G \pi d^4} dP. \end{aligned}$$

The total variation dh_c in the overall height of the spring becomes

$$dh_c = dP \frac{32}{G \pi d^4} \int_0^{2\pi n} R^3 \cos \beta \cos \delta d\varphi, \quad (5.26)$$

where n is the number of turns of the spring, and the quantities $R \cos \beta$ and $\cos \delta$ (and, therefore, the angle α) constitute specific functions of the angle φ .

Exactly the same result is obtained by the usual derivation of the deflection of such springs when only the twist of their elements is allowed for.

Usually, however, in view of the fact that $\cos \beta$ and $\cos \delta$ do not ordinarily vary greatly from unity in value, and also because of the fact that a certain additional deflection of the spring is produced by its straightening moment $dPR \sin \beta \cos \delta$, the following simplified formula is usually employed in calculating such springs:

$$dh_c = dP \frac{32}{G \pi d^4} \int_0^{2\pi n} R^3 d\varphi. \quad (5.27)$$

STAT

Additional deflections due to moments of flexure will produce, both in cylin-

drical springs, a certain additional variation of the α angles. However, even allowing for this latter circumstance, it may nevertheless be assumed, in first approximation, that deflection of a spring will not change the nature of the phenomenon described in this Section, namely that, on intersection of the spring by planes passing through the central axis of the spring (which is the line of action of the load), the diameters of the sections normal to the central axis of the spring remain approximately perpendicular to this axis even after deflection of the spring.

Section 4. Some Notes for Calculating Flexion Springs. Spiral Flexion Springs, and Certain Signs Indicating the Accuracy of the Mathematical Formulas

Formulas for calculating spiral springs (Fig.36) are derived in most literature sources only for cases in which the ends A and B are both fixed (and not merely engaged) in two parts rotatable relative to each other about an axis O. For this case, we obtain a formula for maximum tension

$$\sigma = \frac{M_B}{W}, \quad (5.28)$$

where M_B is an external moment tending to rotate one part relative to the other,

W is the resistance moment of the cross-sectional area of the spring. The equation for the angle of rotation of the two parts relative to each other $\Delta\gamma$ has the form:

$$\Delta\gamma = \frac{M_B L}{EJ}, \quad (5.29)$$

where L is the developed length of the spring;

J is the moment of inertia of the cross-sectional area of the spring;

E is the modulus of elasticity of the spring material.

It is clear from these formulas that an identical bending moment is assumed STAT all points in the spring. Actually, if the bending moment M_1 is not identical at

all points in the spring and is related in some functional manner to the position of the spring component on which it acts, then, obviously

$$\Delta\gamma = \frac{1}{EJ} \int M_i dL,$$

the integral being determined for the entire contour of the spring. Therefore, if eq.(5.29) is used for the case in question, M_B in this formula constitutes a quantity representing a mean for the various values of the bending moments M_i . However, in that case eq.(5.28) would not apply since, in order for the maximum tension to be determined, the maximum bending moment rather than the mean value of the bending moment must be used.

This shows that, in examining conditions characterizing the degree of applicability of eqs.(5.28) and (5.29), only the case in which both ends of the spring are

fixed in two parts performing rotatory motions relative to each other need be considered. If the fastening of one end of the spring consists merely in it being meshed or engaged, it is obvious that the bending moments of various spring elements will differ noticeably. This last case, will be briefly discussed below.

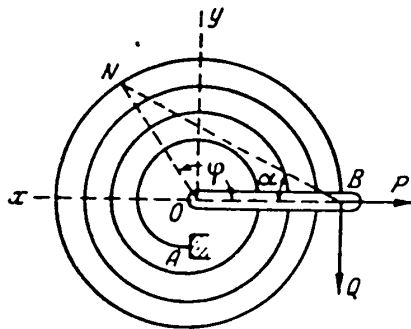


Fig.36 - Study of the Influence of Certain Parameters of a Coil Spring, Working under Flexure, Accurately Calculated by Mathematical Methods

Another indication of the degree of validity of eqs.(5.28) and (5.29) is the following: The axis C of the relative rotation of the two parts in which the ends of the spring are fixed, should, if at all

possible, coincide with the center of gravity of the spring.

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Let one end of the spring be fixed at the point A (Fig.36), which is stationary

relative to the body of the instrument, and the other be engaged to the end B of the lever OB, rotatable about the axis O. Let there be an external moment M_B , rotating the lever OB about the axis O (counteracted by the spring). Since the ends of the spring are fixed, the rotation of the lever through an angle $\Delta\gamma$ will change the total angle of flexure of the spring γ by the angle $\Delta\gamma$, while the distance OB = r between the B end of the spring and the angle of rotation of the lever O remains constant. When this is the case, a bending moment M_B may result at the point of attachment, generating a force that is applied to the end of the spring. This force may be resolved into two components P and Q in the direction of the lever OB and in a direction perpendicular to this.

Let us take a spring segment N of infinitesimal length dL (measuring along the contour of the spiral). In this segment, the bending moment M_1 is

$$M_1 = M_0 - \rho \sin \varphi P + (r - \rho \cos \varphi) Q,$$

where ρ is the distance from the segment N to axis O,

φ is the angle BON.

From this we obtain

$$\Delta(d\varphi) \approx \frac{M_0 - \rho \sin \varphi P + (r - \rho \cos \varphi) Q}{EJ} dL,$$

where $d\varphi$ is the angle between two cross sections of the spring, bounded by the segment N under discussion;

$\Delta(d\varphi)$ is the change in this angle caused by the effect of the bending moment M_1 .

This bending causes the B end of the spring to deflect by an amount dr in the direction OB, which is equal to

$$dr = -\Delta(d\varphi) NB \sin \alpha.$$

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(The overall total change in the distance OB , due to the flexure of all components of the spring, is zero; however, if the bending of a single segment is considered individually, the given change dr would result.) In addition, this flexure induces a displacement $r d\gamma$ of the B end of the spring in a perpendicular direction (in the direction of the force Q), equal to

$$r d\gamma = \Delta(d\varphi) (NB \cos \alpha).$$

From the triangle NOB , we obtain

$$\begin{aligned} NB \sin \alpha &= \rho \sin \varphi; \\ NB \cos \alpha &= OB - \rho \cos \varphi. \end{aligned}$$

Therefore,

$$\begin{aligned} dr &= -\Delta(d\varphi) \rho \sin \varphi = -\frac{M_n - \rho \sin \varphi P + (r - \rho \cos \varphi) Q}{EJ} \rho \sin \varphi dL; \\ r d\gamma &= \Delta(d\varphi) (r - \rho \cos \varphi) = \frac{M_n - \rho \sin \varphi P + (r - \rho \cos \varphi) Q}{EJ} (r - \rho \cos \varphi) dL. \end{aligned}$$

All the spring elements in combination yield the following deflections

$$\left. \begin{aligned} -\Delta r &= 0 = \frac{M_n}{EJ} \int_A^B \rho \sin \varphi dL - \frac{P}{EJ} \int_A^B \rho^2 \sin^2 \varphi dL + \frac{Qr}{EJ} \int_A^B \rho \sin \varphi dL - \\ &\quad - \frac{Q}{EJ} \int_A^B \rho^2 \sin \varphi \cos \varphi dL; \\ r \Delta \gamma &= \frac{M_n r}{EJ} \int_A^B dL - \frac{M_n}{EJ} \int_A^B \rho \cos \varphi dL - \frac{Pr}{EJ} \int_A^B \rho \sin \varphi dL + \\ &\quad + \frac{P}{EJ} \int_A^B \rho^2 \sin \varphi \cos \varphi dL + \frac{Qr^2}{EJ} \int_A^B dL - 2 \frac{Qr}{EJ} \int_A^B \rho \cos \varphi dL + \frac{Q}{EL} \int_A^B \rho^2 \cos^2 \varphi dL. \end{aligned} \right\} \quad (5.30)$$

Since the ends of the spring are fixed, we have

$$\Delta\gamma = \int_A^B \Delta(d\varphi) = \frac{M_B}{EJ} \int_A^B dL - \frac{P}{EJ} \int_A^B \rho \sin \varphi dL + \frac{Qr}{EJ} \int_A^B dL - \frac{Q}{EJ} \int_A^B \rho \cos \varphi dL. \quad (5.31)$$

Multiplying this latter equation by r , and subtracting it from the second equation in the system (5.30), we obtain

$$-\frac{M_B}{EJ} \int_A^B \rho \cos \varphi dL - \frac{P}{EJ} \int_A^B \rho^2 \sin \varphi \cos \varphi dL + \frac{Qr}{EJ} \int_A^B \rho \cos \varphi dL - \frac{Q}{EJ} \int_A^B \rho^2 \cos^2 \varphi dL = 0.$$

Note: The approximations allowed in all the above derivations are, of course, more or less permissible only if the angle of rotation $\Delta\gamma$ of the lever OB is not large. Naturally, if this were not the case it would be impossible to define which quantities ρ and φ to use for each individual segment, since these values vary noticeably when the spring is deflected.

Thus we have three equations permitting a solution for the three unknowns M_B , P , and Q , corresponding to a rotation of the lever OB through some angle $\Delta\gamma$:

$$\left. \begin{aligned} M_B \int_A^B dL - P \int_A^B \rho \sin \varphi dL + Q \left[r \int_A^B dL - \int_A^B \rho \cos \varphi dL \right] &= EJ \Delta\gamma; \\ M_B \int_A^B \rho \sin \varphi dL - P \int_A^B \rho^2 \sin^2 \varphi dL + Q \left[r \int_A^B \rho \sin \varphi dL - \right. \\ &\quad \left. - \int_A^B \rho^2 \sin \varphi \cos \varphi dL \right] = 0; \\ M_B \int_A^B \rho \cos \varphi dL - P \int_A^B \rho^2 \sin \varphi \cos \varphi dL + \\ &\quad + Q \left[r \int_A^B \rho \cos \varphi dL - \int_A^B \rho^2 \cos^2 \varphi dL \right] = 0. \end{aligned} \right\} \quad (5.32)$$

In order for eqs.(5.28) and (5.29) to be valid, it is necessary (as noted above) that the bending moment M_1 is identical at all segments of the spring, i.e., it is essential that the force $P = 0$ and $Q = 0$. We see from the second and third equations in the system (5.32) that, if $P = 0$ and $Q = 0$, we will have

$$\int_A^B \rho \sin \varphi dL = 0 \quad \text{and} \quad \int_A^B \rho \cos \varphi dL = 0,$$

since, when this is the case, the magnitude of M_P will, according to the first equation in (5.32), not be equal to zero.

Conversely, if $\int_A^B \rho \sin \varphi dL = 0$ and $\int_A^B \rho \cos \varphi dL = 0$, then, in accordance with the equations in (5.23), $P = 0$ and $Q = 0$. Thus, this system will yield

$$P = EJ\Delta\gamma \frac{\int_A^B \rho \sin \varphi dL \int_A^B \rho^2 \cos^2 \varphi dL - \int_A^B \rho \cos \varphi dL \int_A^B \rho^2 \sin \varphi \cos \varphi dL}{D};$$

$$Q = EJ\Delta\gamma \frac{\int_A^B \rho \cos \varphi dL \int_A^B \rho^2 \sin^2 \varphi dL - \int_A^B \rho \sin \varphi dL \int_A^B \rho^2 \sin \varphi \cos \varphi dL}{D},$$

where D is the common determinant of the system, i.e.,

$$D = \begin{vmatrix} \int_A^B dL & -\int_A^B \rho \sin \varphi dL & r \int_A^B dL & -\int_A^B \rho \cos \varphi dL \\ \int_A^B \rho \sin \varphi dL & -\int_A^B \rho^2 \sin^2 \varphi dL & r \int_A^B \rho \sin \varphi dL & -\int_A^B \rho^2 \sin \varphi \cos \varphi dL \\ \int_A^B \rho \cos \varphi dL & -\int_A^B \rho^2 \sin \varphi \cos \varphi dL & r \int_A^B \rho \cos \varphi dL & -\int_A^B \rho^2 \cos^2 \varphi dL \end{vmatrix}.$$

Thus, in order for eqs. (5.28) and (5.29) to be more or less applicable, it is necessary for the quantities $\int_A^B \rho \sin \varphi dL$ and $\int_A^B \rho \cos \varphi dL$ to be as close as possible to zero.

This rule may readily be illustrated: For this purpose, let us make use of a rectangular system of coordinates, taking as the Ox axis the direction opposite to that of the lever OB , and as the Oy axis the direction perpendicular thereto. Then, the term $-\rho \cos \varphi$ is the x coordinate of the component under examination, and $\rho \sin \varphi$ is the y coordinate. Thus it is necessary that

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$$\int_A^B x dL = 0 \text{ and } \int_A^B y dL = 0.$$

The expressions on the left sides of both these equations may be presented in the form

$$\int_A^B x dL = L \frac{\int_A^B x dL}{L} = L\bar{X};$$

$$\int_A^B y dL = L \frac{\int_A^B y dL}{L} = L\bar{Y},$$

where $\bar{X}$ and $\bar{Y}$ are the coordinates of the center of gravity of the spring.

In practice, springs of this type are used when the angles of rotation $\Delta\gamma$ are large, causing the springs to deform, with the result that their centers of gravity deviate considerably from the axis 0, even if they coincide with the axis 0 at some position of the spring. Therefore, eqs.(5.28) and (5.29) are even less accurate in many practical cases than the formulas used for calculating other springs (which, as we know, are also somewhat approximated).

Nevertheless, these formulas can be used in a number of cases, but it must be remembered that the spring design must be such as to cause the center of gravity of the spring to be as close as possible to the 0 axis.

For a case in which one end of the spring (end B, for example) is not fixed to the lever OB but is merely engaged with some link at the B end of the lever, this obviously represents a special example of a more general case in which some load Q is applied to a profile of the spring, where the end of the spring is capable of being deflected only in a given specific direction, due to the presence of guides (i.e., due to the action, on the spring, of the reaction force of the guides, in addition to the load P). STAT

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For the case in question, we can use the above method of calculation. Here we have the force Q , applied at the end of a spring normal to the lever OB (see Fig.36), but now the B end of the spring is not fixed, and therefore no bending moment M_1 is at work. Under the effect of this force Q , the B end of the spring should, as we know, undergo a displacement $(f_y)_Q$ in the direction of the axis Oy , of a magnitude

$$(f_y)_Q = -\frac{Q}{EJ} \int_A^B (x+r)^2 dL.$$

and a displacement $(f_x)_P$ in the direction of the Ox axis, equal to

$$(f_x)_Q = \frac{Q}{EJ} \int_A^B y(x+r) dL.$$

However, the fact that the length $OB = r$ cannot change is analogous to the condition that there must be a guide permitting the end of the spring to be displaced only in the direction Oy , normal to the lever OB . Thus, along the lever OB a reaction force P develops, preventing the spring from traveling in the direction Ox . Under the effect of this force, the end of the spring must be displaced by $(f_x)_P$ in the direction of the axis Ox , which is equal to

$$(f_x)_P = -\frac{P}{EJ} \int_A^B y^2 dL$$

and must be displaced by $(f_y)_P$ in the direction of the axis Oy , which is equal to

$$(f_y)_P = \frac{P}{EJ} \int_A^B y(x+r) dL.$$

Thus we have two equations

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$$\frac{Q}{EJ} \int_A^B y(x+r) dL - \frac{P}{EJ} \int_A^B y^2 dL = 0;$$

$$\frac{Q}{EJ} \int_A^B (x+r)^2 dL - \frac{P}{EJ} \int_A^B y(x+r) dL = -f_y = r \Delta \gamma, \quad (5.33)$$

where f_y is the final deflection in the given direction Oy . Excluding from these equations the unknown P , we obtain

$$-f_y = \frac{Q}{EJ} \frac{\int_A^B (x+r)^2 dL \int_A^B y^2 dL - \left[\int_A^B y(x+r) dL \right]^2}{\int_A^B y^2 dL} \quad (5.34)$$

and, for the spiral spring under examination (taking deflection $\Delta \gamma$ to be small),

$$\Delta \gamma \approx \frac{M_n}{r^2 EJ} \frac{\int_A^B (x+r)^2 dL \int_A^B y^2 dL - \left[\int_A^B y(x+r) dL \right]^2}{\int_A^B y^2 dL} \quad (5.34a)$$

To calculate the maximum allowable stress σ , we can determine the direction of the resultant forces P and Q . From the first equation in the system (5.33) we see that the tangent angle β between the axis Ox and the line of action of this resultant is equal to $\frac{\int_A^B y^2 dL}{\int_A^B y(x+r) dL}$. Since the integrals are determined here within definite limits (along the entire length of the spring), the ratio of these two integrals is a definite expression Π for the given spring, constituting a specific function of the configuration of the spring. The maximum bending moment $(M_1)_{\max}$ will exist at the point of the spring contour that is most distant from the straight line, along which the given resultant is effective. This point (and, therefore, its coordinates x_1 and y_1) is usually not difficult to find if the configuration of the spring is known. Subsequently, solving for the magnitude

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$$(M_1)_{\max} = Q(x_1 + r) - Py_1 = Q \left[x_1 + r - y_1 \frac{1}{N} \right], \quad (5.35)$$

we find the value for $\sigma = \frac{(M_i)_{\max}}{W}$.

In the special case in which $\int_A^B y(x+r)dL \approx 0$, so that $P \approx 0$, the formulas for calculating the spring [see eqs.(5.34) and (5.35)] are also considerably simplified.

In the case of a spiral spring engages at one end, the maximum stress will obviously be significantly greater than the maximum stress on a spring in which both ends are fixed, assuming equal angles of rotation $\Delta\gamma$ of the OB lever or identical external moments M_B .

Equations (5.34) and (5.35) are clearly not limited in application to spiral springs alone.

Magnitude of Deflection of Various Points in Springs Pending Under the Effect of Concentrated Forces

As we know, where a straight spring fixed at one end is concerned, its deflection f_C at the point C somewhere between the point A at which the spring is fastened and the point B where it is loaded as well as its deflection f_B at the point B where the load P is applied and its deflection f_D at the point D beyond the point B (if we proceed along the spring from the point A), are determined, respectively, by formulas having the following appearance:

$$\left. \begin{aligned} f_C &= \frac{P}{E} \int_0^{l_1} \frac{(L-x)(l_1-x) dx}{J_x}; \\ f_B &= \frac{P}{E} \int_0^L \frac{(L-x)^2 dx}{J_x}; \end{aligned} \right\} \quad (5.36)$$

$$\left. \begin{aligned} f_D &= \frac{P}{E} \int_0^L \frac{(L-x)(l_2-x) dx}{J_x}, \end{aligned} \right\} \quad (5.36)$$

in which L is the distance AB;

l_1 is the distance AC;

l_2 is the distance AD;

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x is the distance measured along the spring from the point A;

J_x is the moment of inertia of the area of the cross section of the spring, which may also be a function of x .

All these formulas are completely analogous to the integrands. At all points in one of the two members to be multiplied, the magnitude x is subtracted from the distance to the point whose deflection is under examination. However, no such analogy occurs within the limits of integration. Therefore, the indicated regularity in the writing of the integrand may even contribute to rendering more difficult the selection of the correct limits of integration in the eqs.(5.36).

Likewise, in the case of a curved spring (Fig.38), if we take, as the origin of

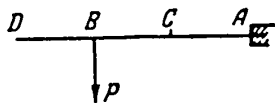


Fig.37 - Straight Spring, Stressed
with Concentrated Load

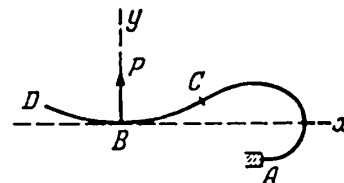


Fig.38 - Curved Spring, Stressed
with Concentrated Load

the coordinates, the point B to which the force P has been applied at some initial position of the spring, plot the Oy axis in the direction of the force P and the Ox axis in the direction normal thereto, then, as we know, the deflections of the points C, B, and D in the direction Oy of the force P will be, respectively,

$$\left. \begin{aligned} (f_C)_y &= \frac{P}{E} \int_C^A \frac{x(x-x_1) ds}{J_s}; \\ (f_B)_y &= \frac{P}{E} \int_B^A \frac{x^2 ds}{J_s}; \\ (f_D)_y &= \frac{P}{E} \int_B^A \frac{x(x-x_2) ds}{J_s}, \end{aligned} \right\}$$

(5.37) STAT

where x_1 and x_2 are, respectively, the x coordinates of the points C and D (coordinates x_2 in Fig.38 being negative, so that the absolute magnitude coincides with that of x);

ds are spring segments of infinitesimal length (measuring on the spring contour;

J_s is the moment of inertia of the cross section of the spring. It may be variable and constitute some function of the coordinates of the given spring element.

Likewise, in the direction Ox , we have the deflections

$$\left. \begin{aligned} (f_C)_x &= -\frac{P}{E} \int_C^A \frac{x(y-y_1) ds}{J_s}; \\ (f_B)_x &= -\frac{P}{E} \int_B^A \frac{xy ds}{J_s}; \\ (f_D)_x &= -\frac{P}{E} \int_B^A \frac{x(y-y_2) ds}{J_s}, \end{aligned} \right\} \quad (5.38)$$

where y_1 and y_2 are, respectively, the coordinates for the points C and D.

Both in the equations of (5.37) and in those of (5.38), the limits of the first integral are the points C and A and those of the second and third are the points B and A in both cases.

Note: We know that it is sometimes necessary to calculate the degree of deflection of a spring in a direction normal to that of load action. For example, the springs of aneroid barometers are bent in a specific curve (Fig.39). It is commonly known that the purpose of this is to cause the spring, when stressed at the point B, to deflect almost solely in the direction of the load (normal to the plane of the diaphragm) and to prevent deflection in a direction normal to the load. This is due to the fact that

vacuum boxes function considerably less satisfactorily (in the sense of error due to hysteresis) at their points of curvature. Therefore, in calculating these springs their profiles are set so that $\int xy \, ds = 0$ at the mean working position (deflection) of the spring.

Thus, a curved spring also presents the same complete analogy in the expression for the integrand and the same violation of this analogy within the limits of integration.

In the case of a straight spring, the stress σ_x developed at a cross section of the spring at a distance x from its attachment point, will reach a magnitude of

$$\sigma_x = \frac{P(L-x)}{W_x},$$

where W_x is the so-called moment of resistance of the section.

In the case of a flat rectangular or trapezoidal spring (the types most frequently encountered), maximum stresses will occur at the point of attachment, i.e., when $x = 0$; only in the case of a flat spring triangular in shape (i.e., if the width of the spring at the point of stress is converted to zero), will we obtain identical stresses in all sections of the spring.

The rather numerous applications of eqs.(5.36) and (5.37), comprise also the derivation of the formulas for the deflection and elasticity of a straight spring fixed at one end, stressed at the other and stopped by a support C at a single point between the point of stressing and the point of attachment (Fig.40). This system is sometimes used in devices where the zero point of the end of the spring is to be controllable (in certain regulators, for example).

Let us denote by f the deflection of the B end of the spring, calculating from the position of this end, for the case that the spring is subject neither to the STAT force P at the B end nor to the force Q with which the stop C acts on the spring in the opposite direction. Further, let us designate as a the distance from the sup-

port C to the spring in the same free position, when it is subject neither to the force P nor to the force Q. This being the case, let us consider the magnitude of a to be positive in the same direction as that of f, i.e., in a case in which the support is already pressing on the spring, but the stress P is not yet present (as often occurs in such designs), the magnitude of a will be negative. Then, in accord-

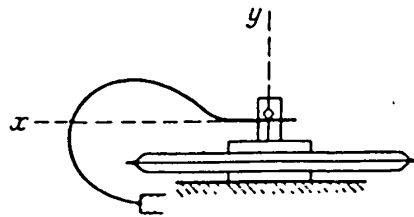


Fig.39 - Typical Form of Curved Spring, Connected to the Block of an Aneroid Cell

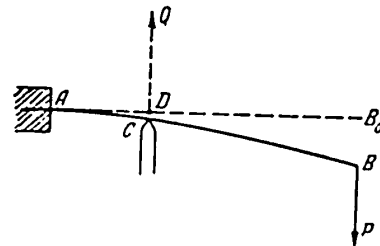


Fig.40 - Straight Spring, Resting on Stop

ance with the principle of the independence of action of forces, eqs.(5.36) will yield

$$\left. \begin{aligned} f &= \frac{P}{E} \int_0^l \frac{(L-x)^2 dx}{J_x} - \frac{Q}{E} \int_0^l \frac{(L-x)(l-x) dx}{J_x}; \\ a &= \frac{P}{E} \int_0^l \frac{(L-x)(l-x) dx}{J_x} - \frac{Q}{E} \int_0^l \frac{(l-x)^2 dx}{J_x}. \end{aligned} \right\} \quad (5.39)$$

In the special case in which $J = \text{const}$, we have

$$\left. \begin{aligned} f &= \frac{P}{EJ} \frac{l^3}{3} - \frac{Q}{EJ} \frac{l^2}{2} \left(L - \frac{l}{3} \right); \\ a &= \frac{P}{EJ} \frac{l^2}{2} \left(L - \frac{l}{3} \right) - \frac{Q}{EJ} \frac{l^3}{3}. \end{aligned} \right\} \quad (5.39a) \text{ STAT}$$

Eliminating the force Q from the two expressions (5.39), we obtain

$$f \int_0^l \frac{(l-x)^2 dx}{J_x} - a \int_0^l \frac{(L-x)(l-x) dx}{J_x} =$$

$$= \frac{P}{E} \left\{ \int_0^L \frac{(L-x)^2 dx}{J_x} \int_0^l \frac{(l-x)^2 dx}{J_x} - \left[\int_0^l \frac{(L-x)(l-x) dx}{J_x} \right]^2 \right\}. \quad (5.40)$$

If the quantity a is negative, i.e., if the stop rests on the spring when the stress P is lacking, the value f_0 corresponding to the initial position of the point B (when $P = 0$) is determined from the equation

$$f_0 \int_0^l \frac{(l-x)^2 dx}{J_x} = a \int_0^l \frac{(L-x)(l-x) dx}{J_x},$$

Therefore, the deflection of the spring, $f_1 = f - f_0$, from this initial position is equal to

$$f_1 = \frac{P}{E} \frac{\int_0^L \frac{(L-x)^2 dx}{J_x} \int_0^l \frac{(l-x)^2 dx}{J_x} - \left[\int_0^l \frac{(L-x)(l-x) dx}{J_x} \right]^2}{\int_0^l \frac{(l-x)^2 dx}{J_x}}. \quad (5.41)$$

Thus, the stiffness of a spring both when a is negative and when it is positive (in the latter case, after the force P has attained a magnitude at which the spring is already resting on its stop) is a constant, and equal to the inverse of the magnitude of the coefficient when P is in the right side of eq.(5.41).

In the special case in which $J = \text{const}$, we have

$$f = \frac{P}{EJ} \frac{\frac{L^3}{3} \frac{l^3}{3} - \left[\frac{l^2(3L-l)}{6} \right]^2}{\frac{l^3}{3}} + a \frac{l^2(3L-l)}{\frac{l^3}{3}} =$$

$$= \frac{P}{EJ} \left[\frac{L^3}{3} - \frac{l(3L-l)^2}{12} \right] + a \frac{3L-l}{2l} = \frac{P}{EJ} \frac{(4L-l)(L-l)^2}{12} + a \frac{3L-l}{2l} \quad (5.42)$$

STAT

and

$$\frac{dP}{df} = \frac{12 EJ}{(4L-l)(L-l)^2}.$$

For example, if $L = 40$ mm, $l = 16$ mm, and a rectangular spring 10 mm wide and 0.6 mm thick is involved, the stiffness of the spring when it rests on the stop, is equal to

$$\frac{dP}{df} = \frac{12 \cdot 2 \cdot 10^6 \cdot 1.8^3 \cdot 10^{-6}}{12(160 - 16)24^2 \cdot 10^{-3}} = \frac{1000}{81} \text{ kg/cm},$$

while in a case in which we had a spring 24 mm long, the stiffness would be

$$\frac{dP}{df} = \frac{3 \cdot 2 \cdot 10^6 \cdot 8^3 \cdot 10^{-6}}{12 \cdot 24^3 \cdot 10^{-3}} = \frac{1000}{54} \text{ kg/cm},$$

i.e., a significantly greater magnitude than that obtained from eq.(5.42).

In the case of a flat spring of rectangular or trapezoidal form, the maximum stresses result either at the cross section resting on the stop, or at the attachment point of the spring, since, in this case, the maximum stresses in the segment between these two sections constitute a monotonic function of the distance x of the spring cross section at the attachment point. In actuality, the bending moment at the given cross section is equal to $P(L - x) - Q(l - x)$, and the moment of resistance W_x becomes

$$W_x = \frac{h^2(b_1 - nx)}{6};$$

where h is the thickness of the spring;

b_1 is the width of the spring at the point of attachment;

$n = \frac{b_1 - b_2}{l}$, where b_2 is the width of the spring at the point of contact with the support.

STAT

Therefore,

$$\sigma = \frac{6[P(L-x) - Q(l-x)]}{h^2(b_1 - nx)};$$

$$\frac{d\sigma}{dx} = \frac{6\{-(P-Q)(b_1 - nx) - n[PL - Ql - x(P-Q)]\}}{h^2(b_1 - nx)^2} =$$

$$= \frac{6[n(PL - Ql) - b_1(P-Q)]}{h^2(b_1 - nx)^2}.$$

Thus, $\frac{d\sigma}{dx}$ does not convert to zero at any point in this segment.

Of course, in order to determine stresses at the two mentioned cross sections, one of which is under great stress, the quantity Q must be known, which requires that the magnitude a is also known; in other words, the degree to which the support is elevated or depressed must be defined.

Example. Let us suppose that, given the spring parameters indicated, it is specified that the spring must not touch the support until a stress P_1 of 200 gm is reached or, in other words, that a diminished rigidity of

$$\frac{3EJ}{L^3} = \frac{3 \cdot 2 \cdot 10^6 \cdot 83 \cdot 10^{-6}}{12.4^3} = 4 \text{ kg/cm},$$

exists; on further stressing to $P_{\max} = 1 \text{ kg}$, the above stiffness equal to

$$\frac{1000}{81} \text{ kg/cm must be present.}$$

To determine the maximum stress σ , permissible in this case, we must first make a theoretical determination of the value a (which, in actuality, will be practically established since the device is controlled). In accordance with the second equation of the system (5.39a)

$$a = \frac{P_1}{EJ} \frac{(3L-l)^2}{6} = \frac{2 \cdot 10^{-1} \cdot 12(120-16) \cdot 16^2 \cdot 10^{-3}}{2 \cdot 10^6 \cdot 83 \cdot 10^{-6} \cdot 6} = 0.0104 \text{ cm} = 0.104 \text{ mm}$$

(where the deflection f of the end of the spring equals $\frac{0.2}{4} = 0.05 \text{ cm} = 0.5 \text{ mm}$).

From this, and in accordance with the same equation (5.39a) and the indicated STAT magnitude a , we have for $P_{\max} = 1 \text{ kg}$

$$Q_{\max} \frac{l^3}{3} = P_{\max} \frac{(3L-l)l^2}{6} - P_1 \frac{(3L-l)l^2}{6};$$

$$Q_{\max} = (P_{\max} - P_1) \frac{(3L-l)}{2l} = \frac{0,8 \cdot 10^4}{2 \cdot 16} = 2,6 \text{ kg}.$$

Thus, at the point C we have

$$\varepsilon_C = \frac{P_{\max}(L-l)}{W} = \frac{6 \cdot 1 \cdot 24 \cdot 10^{-1}}{1,8^2 \cdot 10^{-4}} = 2250 \text{ kg/cm}^2,$$

and at the point A (the attachment point of the spring) we have

$$\varepsilon_A = \left| \frac{P_{\max}L - Q_{\max}l}{W} \right| = \left| \frac{6(4 - 2,6 \cdot 1,6)}{8^2 \cdot 10^{-4}} \right| = 150 \text{ kg/cm}^2.$$

Position of the Neutral Line and Accurate Expression of Flexure Under the Effect of An External Moment on Spring Segments with Some Initial Curvature

In deriving the formulas for calculating curved flexion springs, it is usually assumed, in first approximation, that the angle $\Delta(d\varphi)$ representing a variation in the initial flexion $d\varphi$ of a spring segment of infinitesimal length dl under the effect of an external moment, M is equal to

$$\Delta(d\varphi) = \frac{M_n dl}{EJ}, \quad (5.43)$$

where E is the modulus of elasticity of the material constituting the spring;

J is the moment of inertia of the cross-sectional area of the spring in terms of the midline of this section.

It must be remembered, however, that this formula was derived for the case of a straight segment having no curvature; in which the cross sections, bounding the segment in question, are parallel. In this case, the neutral layer of the spring will coincide with its middle layer (i.e., the rotation will, as it were, take place at the midline of the spring cross section).

However, if the spring had a certain curvature at some point H or, in other words, if an angle $d\phi$ existed between the cross sections AB and CD , bounding a spring segment of infinitesimal length, the variation $\Delta(d\phi)$ in this angle would, under the

influence of the moment M differ slightly from the value $\frac{Mdl}{EJ}$, and the neutral line of intersection will not coincide with the midline of the section. However, the deviation of the angle $\Delta(d\phi)$ from the value of $\frac{Mdl}{EJ}$ and the failure of the neutral line of the cross section to coincide with its midsection, is usually so insignificant that it may be ignored without serious error.

Nevertheless, it must be possible to estimate the magnitude of this error, since it might have some significance in certain special cases.

Let us assume that a bending moment M is applied to some segment of a spring, bounded by two cross sections AB and CD

(Fig.41). Let the segment in question be curved, and the radius of curvature of the center layer EF of the spring (dl long) be equal to R . Let us designate the thickness of the spring by h . Due to the curvature of the spring, its external surfaces AC and BD will differ in length; we will designate the lengths as a and b (due to the infinitesimal length of dl , the lines AC , BD , and EF may be taken as virtually straight). In this case, the length of any layer of the spring x , at a given distance from the external surface BD , may be expressed as the sum $b + kx$, where k is a constant equal to $\frac{a-b}{h}$.

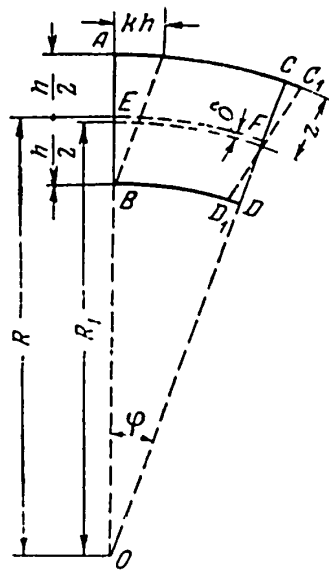


Fig.41 - See Derivation of the Equation for Flexion of a Spring Segment with Initial Curvature

As in other analogous derivations, we will assume that the only moment applied to the spring segment is one of flexion, and that no other forces (such as tension or compression) are applied. Under the influence of the moment M , the cross sections AB and CD start rotating with respect to each other. This being the case, certain layers of the segment become elongated, and others become shortened. Tensile and compressive stresses are generated in the spring material. These stresses may, as an entity, counterbalance the moment M but create no other stresses, since we have agreed that no forces are to act on the segment other than the moment M . If, after bending, the surface AC is of a length a_1 and the surface DB of a length b_1 , then the length of any layer of the spring segment can be expressed in the following form:

$$b_1 + k_1 x, \text{ where } k_1 = \frac{a_1 - b_1}{h}.$$

In this case, the tension in the layer will be

$$E \frac{b_1 + k_1 x - b - kx}{b + kx},$$

where E is the modulus of elasticity of the material.

The resultant stress dP then will be

$$dP = E \frac{(b_1 - b) + (k_1 - k)x}{b + kx} c dx,$$

where c is the width of the spring;

dx is the thickness of the layer.

The sum of all the stresses will become

$$\sum (dP) = Ec \int_0^h \frac{(b_1 - b) + (k_1 - k)x}{b + kx} dx.$$

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As previously indicated, this sum should be equal to zero.

Thus,

$$\int_0^h \frac{(b_1 - b) + (k_1 - k)x}{b + kx} dx = 0; \quad (5.44)$$

$$\int_a^{a_1} \frac{(b_1 - b) + (k_1 - k)x}{b + kx} dx = \frac{k_1 - k}{k} h - \frac{bk_1 - kb_1}{k^2} \ln \frac{b + kh}{b}.$$

Therefore, in accordance with eq.(5.44), we have

$$k_1 - k = \frac{bk_1 - kb_1}{kh} \ln \frac{b + kh}{b}.$$

From this equation, after substituting k and k_1 in the expressions $\frac{a - b}{h}$ and $\frac{a_1 - b_1}{h}$, we obtain

$$(a_1 - a) - (b_1 - b) = \left[\frac{b}{kh} (a_1 - a) - \frac{a}{kh} (b_1 - b) \right] \ln \frac{a}{b},$$

and from this

$$\frac{a_1 - a}{b - b_1} = \frac{\frac{a}{kh} \ln \frac{a}{b} - 1}{1 - \frac{b}{kh} \ln \frac{a}{b}}.$$

From Fig.41 it is evident, however, that

$$\frac{a}{kh} = \frac{R + \frac{h}{2}}{h}; \quad \frac{b}{kh} = \frac{R - \frac{h}{2}}{h}; \quad \frac{a}{b} = \frac{R + \frac{h}{2}}{R - \frac{h}{2}}.$$

From this, we have

$$\frac{a_1 - a}{b - b_1} = \frac{\left(R + \frac{h}{2} \right) \ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}} - h}{h - \left(R - \frac{h}{2} \right) \ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}}}.$$

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The distance z , separating the neutral layer from the external surface AC , is equal to

$$z = \frac{a_1 - a}{k_1 - k} = h \frac{a_1 - a}{(a_1 - a) + (b - b_1)} = h \frac{\left(R + \frac{h}{2}\right) \ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}} - h}{h \ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}}}.$$

The distance δ of the neutral layer from the middle layer is equal to

$$\delta = z - \frac{h}{2} = \frac{R \ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}} - h}{\ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}}} = R - \frac{h}{\ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}}}. \quad (5.45)$$

Usually the value δ is quite small. Thus, when $R = 5.5$ mm and $h = 1$ mm (i.e., even when the ratio of R to h is small), we have

$$\delta = 5.5 - \frac{1}{\ln 1.2} \approx 0.015198 \text{ mm},$$

or, in other words, δ is equal to about 1.5% of the thickness of h .

When the $\frac{R}{h}$ ratio is greater, the magnitude of the $\frac{\delta}{h}$ ratio becomes still smaller.

Now let us determine the magnitude of the angle $\Delta(\varphi)$, by which the sections AB and CD , bounding the spring segment, rotate relative to each other under the effect of the moment M_B .

In order for the section AB to be at equilibrium, it is necessary that the sum

of the moments of stress in all layers of the spring, relative to a given point, be equal and opposite to the external moment M_1 . Let us take the sum of the moments relative to the point O as the center of curvature of the spring (see Fig.41). Let us denote by y the distance of the layer of the spring from the neutral layer, taking y to be positive in the direction to the surface AC. Then, the layer lagging by a distance y from the neutral layer will stretch by the magnitude $(k_1 - k)y$.

This being the case, a tension σ will develop in the layer, equal to

$$\sigma = E \frac{(k_1 - k)y}{dl_1 + ky},$$

where dl_1 is the length of the neutral stratum. This will generate a stress dP in the layer, equal to

$$dP = E \frac{(k_1 - k)y}{dl_1 + ky} c dy,$$

where c is the width of the spring;

dy is the thickness of the layer.

The moment dM of this stress, relative to the point O, is equal to

$$dM = E \frac{(k_1 - k)y}{dl_1 + ky} c dy (R_1 + y),$$

where R_1 is the distance of the neutral layer from the point O.

However, Fig.41 indicates that

$$\frac{R_1 + y}{dl_1 + ky} = \frac{R_1}{dl_1} = \frac{R}{dl},$$

where R is the radius of curvature of the center section of the spring;

dl is the length of the same center section.

From this, we may express the moment of stress as follows:

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$$dM = Ec \frac{R}{dl} (k_1 - k) y dy,$$

or

$$dM = Ec \frac{R_1}{dl_1} (k_1 - k) y dy.$$

The total moment M which must be equal to the external moment M_B , is determined from the following formula

$$M = Ec \frac{R}{dl} (k_1 - k) \int_{-\frac{h}{2} + \delta}^{\frac{h}{2} + \delta} y dy = Ec \frac{R}{dl} (k_1 - k) h \delta,$$

from which

$$k_1 - k = \Delta(d\varphi) = \frac{M_B dl}{Ec R h \delta}, \quad (5.46)$$

or

$$\Delta(d\varphi) = \frac{M_B dl_1}{Ec R_1 h \delta}.$$

For example, for the previous case ($R = 5.5$ mm and $h = 1$ mm) we have

$$\frac{M_B dl}{EJ} = \frac{M_B dl}{Ec} : \frac{h^3}{12} \approx \frac{M_B dl}{Ec} : 0.083333$$

and

$$\frac{M_B dl}{Ec R h \delta} = \frac{M_B dl}{Ec} : (R h \delta) = \frac{M_B dl}{Ec} : (5.5 \cdot 1 \cdot 0.015138) \approx \frac{M_B dl}{Ec} : 0.08327,$$

or, in other words, a difference of less than 0.1% will be obtained.

It is readily shown that eq.(5.46) changes into eq.(5.43) in a case in which the ratio $\frac{R}{h}$ reaches infinity (corresponding to the case of a straight spring); ^{STAT} this is due to the fact that if the ratio $\frac{R}{h}$, increasing gradually, tends toward zero, the derivative $R\delta$ in the denominator (5.46), tends toward a value of $\frac{h^2}{12}$. In

actuality,

$$\begin{aligned}
 R^2 &= \frac{\left[\ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}} - \frac{h}{R} \right]}{\ln \frac{R + \frac{h}{2}}{R - \frac{h}{2}}} = \frac{\left[\ln \left(\frac{\frac{2R}{h} + 1}{\frac{2R}{h} - 1} \right) - \frac{h}{R} \right]}{\ln \left(\frac{\frac{2R}{h} + 1}{\frac{2R}{h} - 1} \right)} \\
 &= \frac{R^2 \left\{ 2 \left[\frac{h}{2R} + \frac{1}{3} \left(\frac{h}{2R} \right)^3 + \frac{1}{5} \left(\frac{h}{2R} \right)^5 + \dots \right] - \frac{h}{R} \right\}}{2 \left[\frac{h}{2R} + \frac{1}{3} \left(\frac{h}{2R} \right)^3 + \frac{1}{5} \left(\frac{h}{2R} \right)^5 + \dots \right]} \\
 &= \frac{R^2 \left[\frac{1}{3} \left(\frac{h}{2R} \right)^3 + \frac{1}{5} \left(\frac{h}{2R} \right)^5 + \dots \right]}{2 \left[\frac{h}{2R} + \frac{1}{3} \left(\frac{h}{2R} \right)^3 + \frac{1}{5} \left(\frac{h}{2R} \right)^5 + \dots \right]} \\
 &= \frac{R^2 \frac{1}{3} \left(\frac{h}{2R} \right)^3 \left\{ 1 + 3 \left[\frac{1}{5} \left(\frac{h}{2R} \right)^2 + \frac{1}{7} \left(\frac{h}{2R} \right)^4 + \dots \right] \right\}}{\frac{h}{2R} \left[1 + \frac{1}{3} \left(\frac{h}{2R} \right)^2 + \frac{1}{5} \left(\frac{h}{2R} \right)^4 + \dots \right]} \\
 &= \frac{h^2}{12} \frac{\left[1 + \frac{3}{5} \left(\frac{h}{2R} \right)^2 + \frac{3}{7} \left(\frac{h}{2R} \right)^4 + \dots \right]}{\left[1 + \frac{1}{3} \left(\frac{h}{2R} \right)^2 + \frac{1}{5} \left(\frac{h}{2R} \right)^4 + \dots \right]}
 \end{aligned}$$

It is quite obvious that this expression tends toward $\frac{h^2}{12}$ as its limit, if the ratio $\frac{h}{2R}$ tends toward zero.

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CHAPTER VI

SPRING SEGMENTS OF VARIABLE STIFFNESS

Section 1. Diaphragms and Diaphragm Cells

Diaphragms and cells consisting of diaphragms soldered together are widely employed in measuring pressure.

In the majority of cases, corrugated diaphragms are used since they have a number of advantages. They have a greater maximum deflection at smaller stresses in the material than do flat membranes, as well as a straighter characteristic curve (i.e., a less pronounced change in stiffness at various deflections of the diaphragm). The use of a corrugated diaphragm, to some degree, permits changing its characteristic in the desired direction by selecting corrugations of the desired shape and depth.

While corrugation has the effect, to some degree, of reducing the stress produced by the general stretching of the diaphragm due to its deflection, it tends to increase the stress produced by bending (particularly in annular directions, i.e., on rotation of radii). Therefore, at small deflections under the effect of low pressures, when the tension due to bending is still the predominant force in the diaphragm, corrugated diaphragms usually have a somewhat greater stiffness than flat diaphragms of the same thickness. When deflection is great and the predominant tensions are those due to stretching of the outer section of the diaphragm, the stiffness of a corrugated diaphragm is usually lower than that of a flat one. Therefore, in many cases, the total deflection of a corrugated diaphragm under maximum load is greater than that of a flat diaphragm under the same load. Thus, corrugated dia-

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phragms have a straighter characteristic curve (ratio of deflection to pressure applied) than flat ones.

Experimental data and theoretical research indicate that the ratio of the pressure p acting on a diaphragm and its deflection W_0 (at the center of the diaphragm) are readily approximated, in virtually all diaphragms, by expressions of the type

$$\frac{p}{E} = A \left(\frac{W_0}{h} \right) + B \left(\frac{W_0}{h} \right)^2 + C \left(\frac{W_0}{h} \right)^3, \quad (6.1)$$

where E is the modulus of elasticity of the diaphragm material;

h is the thickness of the diaphragm;

A, B and C are dimensionless coefficients, constituting functions of the thickness of the diaphragm h , its radius R , the type and depth of corrugation, the number of corrugations, etc.

In formulas for flat diaphragms, the term $B \left(\frac{W_0}{h} \right)^2$ is absent. The presence of this term in many corrugated diaphragms shows that their deflection under the action of pressure in one direction is not identical with the deflection in the opposite direction, since this term does not change its sign on changes in p and W_0 . Usually, however, the term $B \left(\frac{W_0}{h} \right)^2$ is quite small relative to the two other terms on the right side of eq.(6.1).

Theoretical studies of the question lead to insoluble differential equations, even in calculations for flat diaphragms. The calculation of corrugated diaphragms is an even more complex matter. Therefore, all researchers resolve this question in approximated and consequently not very exact fashion. In view of this, the use of equations obtained on the basis of these studies is permissible only for purposes of orientation, to determine the approximate order of magnitude of the deflection of the diaphragm under pressure. Certain of these equations are derived below.

Let us note, to begin with, that the functional connection between the deflection of the diaphragm and the exerted pressure varies slightly, depending on the nature of the attachment of the edges of the diaphragm, i.e., as to whether it is

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sliding (Fig.42) or sealed (Fig.43). Sometimes the seal of a diaphragm cell, consisting of two diaphragms soldered together, may be regarded as sliding.

For flat diaphragms and free attachment of the edges of the diaphragm we have (Bibl.6)

$$\frac{p}{E} = \frac{h^4}{R^4} \left[\frac{16}{3(1-\mu^2)} \frac{W_0}{h} + \frac{6}{7} \left(\frac{W_0}{h} \right)^3 \right], \quad (6.2)$$

and with sealed attachment

$$\frac{p}{E} = \frac{h^4}{R^4} \left[\frac{16}{3(1-\mu^2)} \frac{W_0}{h} + \frac{2}{21} \frac{23-9\mu}{1-\mu^2} \left(\frac{W_0}{h} \right)^3 \right].$$

In these equations, μ is Poisson's modulus.

The use of these equations is recommended only when the deflection of the dia-

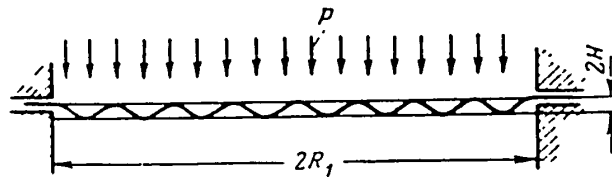


Fig.42 - Diaphragm with Sinusoidal Corrugations and Sliding Seal

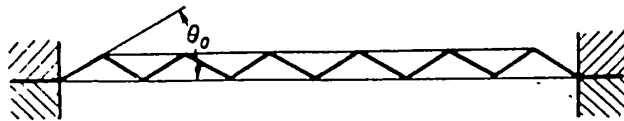


Fig.43 - Diaphragm with Sawtooth Corrugations and Blind Seal

phragm W_0 is small, approximately at the level of deflections of four times the thickness of the diaphragm. On further increase in deflection, the factors operative at $\left(\frac{W_0}{h} \right)^3$ should be somewhat increased and may, at high deflections, reach 3.4.

For corrugated diaphragms with sinusoidal corrugations, the cited book by V.I. Feodos'yev gives the equation

$$\frac{p}{E} = \frac{h^4}{R^4} \left\{ \left[A_1 \frac{H^2}{h^2} + \frac{16}{3(1-\mu^2)} \right] \left(\frac{W_0}{h} \right) - A_2 \frac{H}{h} \left(\frac{W_0}{h} \right)^2 + A_3 \left(\frac{W_0}{h} \right)^3 \right\}, \quad (6.3)$$

where H is the thickness of the corrugation

$$\begin{aligned} A_1 &= -96 \alpha^2 m_1; \\ A_2 &= 24 \cdot 16 \alpha m_2; \\ A_3 &= -24 \cdot 64 m_3. \end{aligned}$$

Here $\alpha = 2\pi \frac{R}{l}$, where l is the length of a full corrugation (i.e., the length of a full sinusoid of corrugation).

Further,

$$m_3 = \frac{1}{96} \left(\frac{1}{14} + \frac{a_1}{24} \right),$$

where $a_1 = -3$ for free sealing and $a_1 = -\frac{5-3\mu}{1-\mu}$ for blind sealing.

The magnitudes m_1 and m_2 are rather complex expressions, constituting functions of several parameters of the diaphragm; the radius of the flat uncorrugated area at the center of the diaphragm, the number of corrugations (the formulas themselves become considerably more complex if the number of corrugations is not an integer), etc. In a case of complete corrugation of the diaphragm (the radius of the flat center being zero) and corrugations totaling a whole number, the expressions for m_1 and m_2 become considerably more complex, being

$$\begin{aligned} m_1 &= \frac{1}{12a^2} - \frac{15}{8a^4} - \frac{105}{8a^6} - \frac{12a_2}{a^3} + \frac{b_2}{a}; \\ m_2 &= \frac{38+12a_1}{96a^3} - \frac{11}{2a^5} + \frac{1575}{a^7} - \frac{56700}{a^9} - \frac{a_2}{21} - \frac{b_2}{6}. \end{aligned}$$

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The quantities a_2 and b_2 in the total expressions for determining m_1 and m_2 are also functions of various parameters of the diaphragm. In the case in question (com-

plete corrugation with the corrugations totaling a whole number), these functions are considerably simpler and become

$$b_2 = \frac{1}{a^3} + \frac{12}{a^5}$$

both for free and for blind sealing of the edges of the diaphragm; for free sealing, the term a_2 is

$$a_2 = -\frac{5}{a^3}$$

and for blind sealing

$$a_2 = \frac{7 + 5\mu}{a^3(1 - \mu)}$$

Here $a = 2 \pi \cdot n$, where n is the number of corrugations.

Thus, for free sealing we obtain, as a result

$$\begin{aligned} A_1 &= 8 \left(1 + \frac{21}{2a^2} - \frac{1413}{2a^4} \right); \\ A_2 &= \frac{24}{a^2} \left(1 - \frac{120}{a^2} + \frac{25200}{a^4} - \frac{907200}{a^6} \right); \\ A_3 &= \frac{6}{7}. \end{aligned}$$

Theoretical formulas for calculation are also known to exist for certain other types of corrugation. For example, for continuous sawtooth corrugations (Fig.43) in which all corrugations are complete, we have, according to the derivations by L.Ye. Andreyeva (Bibl.12)

$$\frac{pR^4}{Eh^4} = \left[\frac{16 \cos^3 \theta_0}{3(1 - \mu^2)} + n_2 \frac{\sin^3 \theta_0}{\cos \theta_0} \frac{R^2}{h^2} \right] \frac{W_0}{h} + n_1 \sin \theta_0 \frac{R}{h} \left(\frac{W_0}{h} \right)^2 + \xi \left(\frac{W_0}{h} \right)^3. \quad (6.4)$$

Here, for a sliding seal, we have

STAT

$$\xi = -\frac{6}{7} \cos \theta_0;$$

$$n_1 = \frac{96.24}{\pi a^3} \left(-0.04228 + \frac{5.007}{a^2} - \frac{1050}{a^4} + \frac{37800}{a^6} \right);$$

$$n_2 = \frac{64.24}{\pi a^2} \left(0.08456 + \frac{0.8345}{a^2} - \frac{60.95}{a^4} \right);$$

and for a blind seal

$$\xi = 2.762 \cos \theta_0.$$

$$n_1 = \frac{96.24}{\pi a^3} \left(0.6825 + \frac{5.007}{a^2} - \frac{1050}{a^4} + \frac{37800}{a^6} \right);$$

$$n_2 = \frac{64.24}{\pi a^2} \left(0.04456 + \frac{0.8345}{a^2} + \frac{150.8}{a^4} \right).$$

We limit ourselves here merely to directing attention to the derived formulas, since they are all highly approximate and are used, in practice, only for purposes of orientation. After the first specimens of diaphragms of one type or another have been made, their characteristics are usually determined experimentally. For purposes of primary tentative selection of the class of diaphragm, one may also make use of the catalogs of diaphragm characteristics issued by certain offices.

Let us note, also, that the class of diaphragm characteristics is greatly influenced by the so-called rim corrugation, i.e., by an enlarged half-corrugation gradually converting to an almost cylindrical flange along the edge of the diaphragm (Fig.44).

The literature also contains certain approximated theoretical formulas allowing for the effect of the rim corrugation (Bibl.13), but we will not derive them here since they are highly approximate.

Above (see Section 3, Chapter I) we indicated precisely what is understood by the term "effective area" of a diaphragm. Another definition of effective area also exists. If the differential pressure on a diaphragm is varied by some small value Δp and the force applied at its center is modified accordingly by a value ΔQ ^{STAT} so that there is no displacement of the center of the diaphragm, the $\frac{\Delta Q}{\Delta p}$ ratio will

equal the effective area of the diaphragm. The derivative $\frac{dQ}{dp}$ offers a more precise concept of effective area.

This area is, of course, smaller than the total area of the membrane. In fact, the force applied to the center of the diaphragm does not counteract the total force derived due to the pressure p on the total area of the diaphragm, since a portion of this latter (more than half, in fact) is applied at the seals of the diaphragm edges.

The magnitude of the effective area of the diaphragm varies somewhat both with

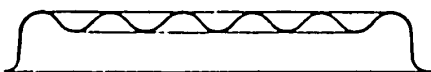


Fig. 14 - Diaphragm with Rim

Corrugation

changes in the deflections or variations in the values of the distributed pressure p and the concentrated load Q , corresponding to an identical deflection of the diaphragm.

In other words, if the total deflection of the diaphragm x_p , due to the distributed load p (the deflection occurring in the absence of an opposing concentrated force Q) is intentionally divided in two

$$x_{p_0} + x_{p_1} = x_p$$

and if we designate the initial deflection of the diaphragm as x_{p_0} due to the load p_0 , while we denote by x_{p_1} the further deflection resulting from the additional pressure p_1 (thus, $p_0 + p_1 = p$), at $x_{p_1} = -x_Q$ where x_Q is the deflection under the effect of a concentrated load Q , then the size of the effective area may change both as a result of the initial deflection x_{p_0} and as a result of the fact that, on further combined loading by the pressure p_1 and the concentrated force Q (in such a fashion that the initial deflection x_{p_0} does not change), the relationship between the pressure p_1 and the force Q will not be precisely linear. Particularly pronounced is the effect of change in the magnitude of p_0 , or in other words, the x_{p_0} deflection.

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According to data of the Leningrad Research Institute, the effective area, S_{eff} may be taken, on the average, as equal to

$$S_{eff} \approx \pi(0,3R^2 + 0,4Rr + 0,3r^2), \quad (6.5)$$

where R is the radius of the diaphragm;

r is the radius of its rigid center.

Here it must be noted, however, that in accordance with the new, more detailed and entirely credible experimental data by N.I. Rutszkaya, engineer, this formula is in many situations more or less correct only when $p_0 = 0$ and the magnitudes of p_1 and Q are relatively small (while, even when $p_0 = 0$, the formula is not correct for all diaphragms). As for cases in which the pressure p_0 and, consequently, the deflection x_{p_0} are not equal to zero, the research cited indicates that, if the diaphragm characteristic is not linear, i.e., if the relationship between the pressure p on the diaphragm and its deflection x is not linear and the magnitudes of p_1 and Q are comparatively small (which is usually the case in calculations of errors due to friction and to unbalance in the parts of the mechanism), the effective area will also show a noticeable reduction with an increase in the deflection x , but this will still be slightly less than the increase in rigidity.

As far as cases are concerned in which the concentrated force Q is greater (i.e., in instruments in which two elastic elements interact under stress) the general picture is analogous. However, this question has been even less studied and needs further investigation, since the following factors will also be involved: first, the types of diaphragms, both in distributed and in concentrated loading, and secondly the degree to which deformation of the diaphragm under a given type of load (producing corresponding variations in stiffness) are analogous to deformations under other types of loading.

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Section 2. Calculation of Deflection of Straight Springs Fixed at One End and Supported at a Number of Points

We know that, to secure springs with gradually increasing stiffness, straight springs which come to rest on supports, upon deflection, are often used. Usually, the stops consist of a series of screws, the height of which may be regulated by turning. Springs of this type are mounted, for example, in twin-pointer airspeed indicators, where the device in question is employed to obtain uniform scaling.

As the spring, under a successive increase in deflection, i.e., under gradual enlargement of the load on its end, comes to rest on a new stop, its stiffness increases accordingly, and in the range of deflections (i.e., in the range of the loads) under which the spring always rests on the same specific stops, its stiffness remains constant; however, this stiffness depends solely on the spring dimensions and the distances l_1, l_2, l_3 , separating the stops on which it is already resting

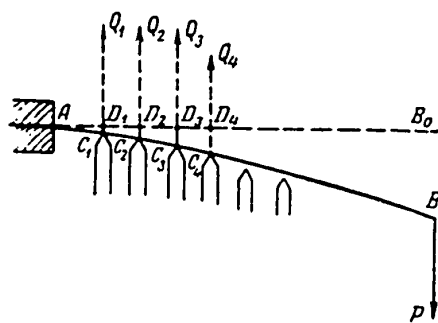


Fig.45 - Spring Supported by Stops
as it Deflects

from the point of attachment of the spring (Fig.45), and is not dependent on the distances C_1D_1, C_2D_2, C_3D_3 , etc., of the ends C_1, C_2, C_3 , etc., of the stops from the spring in its unstressed position AB_0 .

This stiffness is somewhat less than the stiffness of a spring whose length is equal to the free unsupported portion of the spring, i.e., the "reduced" length of the given spring is somewhat larger than the free unsupported portion of the spring.

This stands to reason, since the spring segments between stops also work in some fashion as the load P on the end of the spring rises, although they do so of course to a considerably lesser degree than when they are free.

As far as the distances C_1D_1, C_2D_2, C_3D_3 , etc. are concerned, they govern only

the load at which the elasticity of the spring changes (since its changes are obviously in leaps).

If the relationship between the load P and the deflection f is presented in graphic form (Fig.46), the result is a broken line consisting of linear segments. The magnitude of the loads P_1 , P_2 , etc., at which the slope of the straight lines changes, depends on the quantities C_1D_1 , C_2D_2 etc., while the angles of slope of the individual segments of the broken line are not dependent on these quantities.

The equation for determining the spring elasticity when there is a constant num-

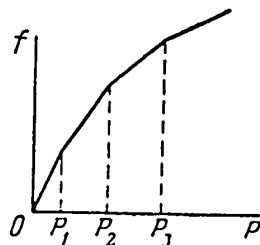


Fig.46 - Ratio of Load to Deflection in a Stop-Supported Spring

ber of identical stops, is readily derived by the same method we used in deriving the elasticity of a spring resting on a single stop. Here we also obtain an analogous formula, although a more complicated one (in accordance with the number of stops).

This derivation is obtained by means of a system of equations analogous to the system for the two expressions in eq.(5.39), but this system will have a correspondingly larger number of equations (larger by one than the number of stops), and the number of terms in each equation will increase accordingly. We assume, in this connection, that each stop on which the spring rests, presses against the spring with a force of Q_1 or Q_2 , etc. Eliminating from these equations the stresses Q_1 , Q_2 , etc., we obtain an equation analogous to that in the system (5.40).

We obtain the above properties for this type of system from the derivation in the same fashion as we obtained analogous properties in deriving the formula for a spring resting on a single support.

Thus, if we assume that, with a gradual increase in the load P , a spring resting on a number of supports will not subsequently be separated therefrom, it may be

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stated that variations in the quantities C_1D_1 , C_2D_2 , etc. result only in a displacement of the segments of the broken line parallel to themselves. Thus, in this case and by a corresponding selection of the quantities C_1D_1 , C_2D_2 , etc., we may inscribe the above-mentioned broken line within virtually any curve that answers the single requirement that $\frac{dP}{df}$ and $\frac{d^2P}{df^2}$ be positive at all times. If the number of supports is adequate (and they are usually designed so as to be of adjustable height), then we are obviously in a position to obtain a broken line sufficiently similar to the given curve.

On the basis of results of investigation of the working of springs resting on curved supports when deflected, we may conclude that situations are fully possible in which a spring resting on stops becomes detached from these stops after it comes to rest on subsequent stops, causing the load P to increase. Obviously, in this case too, a proper adjustment of the height of the stops leads to a situation in which the broken-line characteristic of the spring is adequately inscribed within a curve of the required characteristics. This is not difficult to achieve experimentally, while theoretical investigations of the question as to whether the spring will, in a given case, rise clear of the supports, and determination of the loads at which this will occur, are quite complex.

Note: Theoretical calculations of such springs are also no more than approximate since the nature of the working of the spring is somewhat distorted by the forces of friction between the spring and its supports. However, allowance for these friction forces would, to begin with, greatly complicate the matter, while often it is virtually impossible to make an exact determination of their precise effect. In the second place, these forces respond differently to increases and reductions in the load P , so that, for the most part, they create an effect analogous to a variance in the operation of the spring.

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If we examine only flat springs of constant thickness, rectangular or trapezoidal in cross section, in which the resistant moment W_x is

$$W_x = \frac{h^2 (b_0 - nx)}{6},$$

where h , b_0 , and n are constants, then, in determining the permissible stresses in the spring, the critical cross section is either at A where the spring is mounted, or at one of the points C_1 , C_2 etc. Therefore, it is necessary to calculate only the moments of flexure at these points. This may readily be proved in a fashion analogous to the proof for springs resting on only one support.

However, in order to determine the moments of flexure at the points A , C_1 , C_2 , etc., it is necessary to determine the magnitude of the forces Q_1 , Q_2 etc. at maximum load P_{max} on the end of the spring, and also at loads at which the spring begins to touch each of the successive stops. These forces Q may be determined from the system of equations to which we have referred, analogous to the system (5.39). However, here it should be noted that, in order to do this, we must know the magnitudes C_1D_1 , C_2D_2 etc. since these determine the magnitude of the forces Q_1 , Q_2 , etc. Therefore, calculations of this type are excessively complicated.

It should be noted, further, that even this method of analyzing only the elas-

ticity of a spring resting on a support is clumsy when applied to a rather large number of supports, and that this also complicates the theoretical analysis of such springs to some degree:

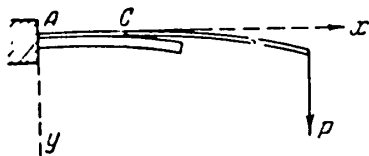


Fig.47 - Spring Resting on a Curved Support, When Deflected

If the supports are spaced closely enough, it is more convenient, practically speaking, to analyze the spring by the general method of calculation for springs resting on curved supports

(Fig.47), taking the supports to approximate a solid curve. In this case it is ^{STAT} even necessary to determine the total profile length of the rest required. All that is needed is to determine the elasticity of the unit at two extreme loadings of the

spring, and to determine whether these elasticities correspond to the required maximum and minimum elasticities.

It should be noted at the outset that the following has been found in more detailed studies of the question of analyzing the profile of a molded support, for springs gradually coming to rest thereon with increase in deflection (Bibl.7).

Let us take as given a certain required ratio of the load P at the end of the spring to the corresponding deflection of its end f , expressed by the equation

$$f=f(P) \quad (6.6)$$

or

$$f-f_0=f(P)-f(P_0),$$

where f_0 is the yield of the spring at an initial stress P_0 , i.e., before the support goes into effect.

Research has shown that when the $f(P)$ function takes on certain forms, a curved support must be patterned in accordance only with the second of the two equations derived. In other words, in an analysis of the spring and the curve, not only its maximum load $P_{\max}$ must be given, but also the minimum load $P_{\min}$ at which the curved support comes into operation.

Depending on the form of the function $f(P)$, the spring and its shaped support may operate in accordance with one of two principles:

1. At a load P , the segment AC of the spring osculates the curve over its entire length, while the remainder of the spring CB does not rest on it at any point.
2. When loaded at the point B, the spring touches the curve only at the point C, i.e., it is not in contact with the curve in the segment AC, while the point C is also gradually displaced along the curve, with any variation in the load P .

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In this respect it appears that, if

$$2 \frac{dP}{dx} - \frac{d^2P}{dx^2} (L-x) > 0, \quad (6.7)$$

where the connection between the quantities P and x is determined by the equation

$$\frac{d[f(P)]}{dP} = \frac{1}{E} \int_x^L \frac{(L-x_1)^2 dx_1}{J_{x_1}},$$

the spring and its curved support function in accordance with the former principle.

If, given the same connection between the quantities P and x , we have

$$2 \frac{dP}{dx} - \frac{d^2P}{dx^2} (L-x) < 0, \quad (6.7a)$$

the spring and its support work in accordance with the latter principle.

In these equations, L is the total length of the spring; x is the distance of the extreme point C of contact between the spring and support, and the root of the spring, under the load P (thus, $L-x$ is the length of the free portion of the spring); x_1 is the distance of the given segment of length dx_1 from the root of the spring; and J_{x_1} is the moment of inertia of the spring cross section at this point.

Thus, the order of calculation of this type of device is as follows:

If the ratio $f = f(P)$ is given in analytic form, then it must first be determined from eq.(6.7) which of the two principles is followed in the operation of the support and spring.

If the spring works in accordance with the first principle, then the minimum stiffness of the spring $c_{\min}$ at which it does not yet rest on the support at any point over its length, is expressed by the equation

$$\left\{ \frac{d[f(P)]}{dP} \right\}_{\max} = \frac{1}{c_{\min}} = \frac{1}{E} \int_0^L \frac{(L-x)^2 dx}{J_x}. \quad (6.8)$$

In the special case in which $J_x = \text{const}$, we have

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$$\frac{1}{c_{\min}} = \frac{L^3}{3EJ}.$$

Using this equation and the given quantity $c_{\min}$, it is possible to plan all the dimensions of the spring itself (a preliminary plan, however, since after subsequent verifying analysis of the maximum stress σ , produced in the spring on maximum loading, it may be necessary to change these dimensions somewhat).

Further, for maximum stiffness of the design $c_{\max}$, we have the equation

$$\left\{ \frac{d[f(P)]}{dP} \right\}_{\min} = \frac{1}{c_{\max}} = \frac{1}{E} \int_1^L \frac{(L-x)^2 dx}{J_x}, \quad (6.9)$$

where l is the x coordinate of the point C when the spring is under maximum load.

In the special case in which $J_x = \text{const}$, we have

$$\frac{1}{c_{\max}} = \frac{(L-l)^3}{3EJ}.$$

From this equation, we determine the magnitude of l from the given magnitude of $c_{\max}$.

We may then determine the maximum stress from the equation

$$\sigma = \frac{P_{\max}(L-l)}{W_{x=l}}. \quad (6.10)$$

If, with the spring dimensions obtained in a preliminary form on the basis of eq.(6.8), we obtain an impermissible value for σ , it is necessary to try a spring satisfying eq.(6.8) with somewhat different dimensions, and then to repeat the check analysis. Prior to the final determination of the dimensions of the entire device it may be necessary to make several attempts in this fashion, but unfortunately the relation here between eqs.(6.8), (6.9), and (6.10) is such that it is impossible to define first certain spring dimensions by the formula for maximum stress and then to determine the remaining dimensions by the formula for the elasticity of the spring, as is often done in calculating somewhat simpler springs.

The magnitude l , determined from eq.(6.9) is the distance from the point of at-

tachment of the spring to the farthest support (most distant from point A) on which the spring rests when deflected. Thus, the supports, adjustable in height, must be mounted in this segment from point A to a point at a distance l from the point A, and preferably as close together as possible.

However, if the device functions in accordance with the second principle, then the minimum stiffness $c_{\min}$ is expressed by the same eq.(6.8) as applies for working in accordance with the first law. Therefore, here, too, we use eq.(6.8) for a preliminary indication of all the dimensions of the spring proper, in accordance with the given value of $c_{\min}$.

However, for the maximum stiffness of the spring $c_{\max}$, we now have another equation

$$\left\{ \frac{d[f(P)]}{dP} \right\}_{\min} = \frac{1}{c_{\max}} =$$

$$= \frac{1}{E} \frac{\int_0^L \frac{(L-x)^2 dx}{J_x} \int_0^l \frac{(l-x)^2 dx}{J_x} - \left[\int_0^l \frac{(L-x)(l-x) dx}{J_x} \right]^2}{\int_0^l \frac{(l-x)^2 dx}{J_x}} \quad (6.11)$$

This equation is derived from eq.(5.41) when only a single support presses against the spring, since in operation of the spring and its curved support in accordance with the second principle (in addition to certain other forces), only the force of pressure of the curved support against the spring at a single point C acts against the root of the spring.

In the special case in which $J_x = \text{const}$, we have

$$\frac{1}{c_{\max}} = \frac{1}{EJ} \frac{\frac{L^3}{9} - \frac{l^3}{4} \left(L - \frac{l}{3} \right)^2}{\frac{l^3}{3}} = \frac{1}{EJ} \frac{(4L-l)(L-l)^2}{12} \quad \text{STAT} \quad (6.11a)$$

Equation (6.11) makes it possible to determine the magnitude of l on the basis of the given magnitude of $c_{\max}$.

Subsequently, by employing eq.(6.10), it is possible to determine the magnitude of the maximum stress σ .

It is clear from these formulas that in cases in which J_x is not a constant, but a function of x , eqs.(6.9) and (6.11) from which the magnitude of l is determined, usually are complex and even analytically unsolvable. However, they are capable of graphic solution, for which purpose it is necessary to plot the curve for $c_{\max}$ on l along a series of points, assigning various values for the magnitude of l .

However, if $J_x = \text{const}$, then eqs.(6.9) and (6.11) may be resolved analytically, while eq.(6.11), corresponding to a case in which the spring works in accordance with the second principle, proves to be a third-order equation relative to l so that eq.(6.9), corresponding to a case in which a spring operates in accordance with the first principle, is quite simple.

In practice, however, the function $f = f(P)$ is often offered not in an analytical form but only graphically, i.e., in the form of a curve presenting the relationship between the load P and the required deflection f of the end of the spring.

Similar situations are possible chiefly in calculations for designs in which the described type of spring works in combination with some other elastic element such as a diaphragm box whose deflection, due to differential pressure, is given not in an analytical form but only in the form of an experimental curve. In such cases, only the relationship between the total load and the total deflection of the entire elastic unit of box and spring is usually given in analytical form. Therefore, to calculate the required spring it is necessary to determine what portion of the total load corresponding to one or another deflection of the total elastic assembly is ascribable to the spring itself.

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If the relationship between the pressure and the deflection of the box (not yet connected to the spring) and also the expression of its effective area at various

deflections and loads were given in analytical form, then it would of course be possible also to derive an analytical expression for the relation of the deflection of a spring to the load thereon. In actuality, in that case the load P on a spring corresponding to a given deflection thereof f is equal to

$$P = (p_c - p_k) (S)_{cp}, \quad (6.12)$$

where p_c is the total pressure which, by definition, must correspond to the total deflection f of the entire spring assembly;

p_k is the pressure at which one box, not connected to the spring, is deflected by a magnitude f ;

$(S_{eff})_{cp}$ is the mean effective area of the diaphragm box on any variation in pressure from p_k to p_c .

However, let us suppose that only the experimental curve for deflection of a diaphragm box has been given. In this case, even if we knew the effective area for any deflection and the loads p and P of the box, the relationship between the load on the spring and its deflection could be only graphically plotted, specifically in the following manner.

Given curve 1 (Fig. 48) for the dependence of the deflection of the box on the pressure acting thereon, let us plot on the same scale the curve 2, for the relation between the total pressure measured by the entire spring assembly and the deflection of this assembly. Thereafter, for each deflection f we may find the corresponding quantities p_c and p_k , in accordance with curves 2 and 1.

To derive the magnitude of the load P , causing the spring to deflect by this amount f , we need only multiply $p_c - p_k$ by the average value of the effective area $(S_{eff})_{cp}$ (in the pressure range from p_k to p_c) of the box. In this manner we may proceed, point by point, to plot the desired ratio of the deflection f of the spring to its load P , but in this case the relationship would no longer be analytical.

Research has shown that, in most diaphragms, the size of the effective area changes very noticeably at various deflections and various combinations of the loads p and P ; therefore, in many cases we do not know these factors for the effective area of the diaphragm.

If, under these conditions, in designing an instrument, we wish to make preliminary

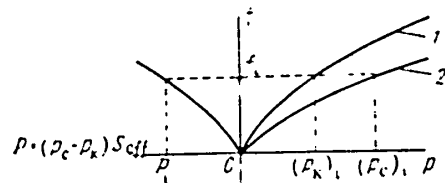


Fig. 4.8 - Plotting the Wanted Relation Between Spring Load and Deflection, Given the Ratios: 1) for the Pressure and Deflection of a Single Box without Spring, and 2) for the Pressure and Deflection of the Total Assembly Consisting of Box and Spring

calculations of the spring and the required length l , it is possible only experimentally to determine the magnitudes P , corresponding to different points in the required characteristic of the entire spring assembly.

Thus, in the majority of cases, even if we prove successful in obtaining the required relationship between the deflection f of a spring and its load P , this relationship is graphic and not analytical.

Under these conditions it is very difficult and often simply impossible to determine by which of the two principles a spring with curved support will function in each specific case, since the use of eq.(6.7) demands the analytical expression $f = f(P)$. Therefore, we can only recommend calculating the spring with a certain factor of safety, as follows:

Without determining which principle the system will follow, we may use as our maximum stress σ , the larger of the figures obtained by the use of the two principles, namely that obtained under the first principle, specifically by means of eq.(6.10), after having first determined the magnitude of l in that equation, in accordance with eq.(6.9). STAT

The final magnitude of the distance l , along which the design will call for the

provision of supports, is determined in accordance with the principle by which it is greatest, i.e., by the second principle, making use of eq.(6.11), since the magnitude of l calculated by the second principle turns is somewhat larger than when calculated by the first. In point of fact, at one and the same magnitude of AC and at identical parameters of the spring itself, its stiffness will be less when the spring works with a curved support in accordance with the second principle, since in the second case the segment AC of the spring also works to some degree.

It should also be noted that when springs capable of variations in stiffness in accordance with any given formula are desired, helical coil-action springs with turns subject to progressive unwinding, are sometimes used.

Such spiral springs may be cylindrical, in which case the lead of the spring is variable, as a result of which the coils of the spring rest on each other when it is loaded to the point of compression.

In addition, springs may be designed with variable turn radii, in conical or other shapes. In this case, the coils, starting with those of greater radius, successively come to rest on the surface on which the spring rests.

However, we do not here derive the formulas for calculating such springs, since they are used quite rarely as pickup elements in serial production of instruments. This is due to the fact that it is impossible to calibrate their characteristics, something that is, conversely, easy to do with straight springs successively coming to rest on supports whose height is subject to adjustment. Persons desiring to do so may familiarize themselves with this question in the studies by Ye.P.Popov (Bibl.8) and Yu.I.Iorish (Bibl.16), and also in a paper by the present author devoted specifically to this subject (Bibl.9).

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CHAPTER VII

COMBINED EFFECTS OF A NUMBER OF POSITIONAL FORCES

Section 1. Reduced Coefficients of Recovery of Positional Forces. Positive and Negative Coefficients of Recovery of Positional Forces

Several forces may often be at work in any given mechanism. They may be applied at various points in the mechanism and act in various directions. We know that in such instances, the total effect of all forces on the mechanism is commonly determined by reducing them to some single point (or axis) of the mechanism, in a single direction.

The forces affecting the mechanism usually include positional forces which may not be brought to bear at the point in the mechanism to which all the forces are reduced, so that it becomes necessary to reduce them as well. Therefore it is also necessary, in many instances, to determine the reduced coefficient of recovery of the reduced positional force.

In certain mechanisms, several positional forces may be active. In some cases they may act at a single point in the mechanism, and in others at different parts. This being the case, it is often necessary to determine the total coefficient of recovery of all the positional forces relative to some single particular point (or axis) of the mechanism. This requires either a simple summation of the coefficients of recovery (if the positional forces are reduced to a single point in the mechanism), or prior reduction of all these coefficients before addition. In the present Chapter we are discussing certain questions related to the reaction of positional

forces, and the addition and reduction of the coefficients of recovery of positional forces.

We have previously indicated exactly what is meant by the terms "positional force" and "coefficient of recovery (or recovery factor) of the positional force" (Chapter II, Section 2). We also indicated that, in the general case, the coefficient of recovery of the positional force may be variable, i.e., it is also some function of the displacement x of the point to which the forces are reduced (only displacement in or opposite to the line of action of the force being considered). Let us, therefore, assume that in an expression presenting the positional force as some function $f(x)$ of the displacement x of the point to which the force is reduced [or in the corresponding plot or Tables, if the $f(x)$ function is given not in analytical form but in graphic or tabular form], we will consider the displacement x to be positive in a direction opposite to that of the force.

By analogy, we call a moment positional when its magnitude depends on the angle of rotation of the part to which it is applied, i.e., when the moment is some function of the angle of rotation of the part. We designate the $\frac{dM}{d\varphi}$ ratio as the coefficient of recovery of the positional moment, where dM is the simple increase in moment corresponding to the simple rotation d of the part in a direction opposite to that in which the moment acts.

Having established the positive and negative directions for measuring the displacement of the points of application of positional forces, we may classify the positional forces on the following principle:

Let the mechanism move in such a fashion that the point of application of a positional force is displaced in a positive direction, i.e., in the direction opposite to that in which the force is applied (the force performing negative work in this respect). Two possible cases exist:

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The force increases when the point of application is displaced as indicated;

The force decreases when the point of application is displaced as indicated.

When the direction of displacement of the points at which the forces are applied is negative, i.e., when the displacement is in the direction in which the force is acting, this force declines in the first instance, and increases in the second.

Thus, if $d|F|$ is used to designate the magnitude of change in the absolute force corresponding to a displacement dx of the point of application, we find that, in the first instance, the expression $\frac{d|F|}{dx}$, i.e., the coefficient of recovery of the positional force, is positive and in the second case negative. Thus, we learn that there are positional forces with positive coefficients of recovery and others with negative coefficients.

It is easy to see that - for example - all springs have positive coefficients of recovery (elasticity), while the centrifugal force, on the other hand, has a negative coefficient. In actuality, the centrifugal force is $m\omega^2 r$, where m is the mass of the body, r the radius of the circle through which the body moves, and ω the angular velocity of rotation. The direction of the centrifugal force is radial, from the center of rotation. This means that an increase in the radius r by a value dr , related to an increase in the centrifugal force, is, in accordance with our definition, a negative displacement dx of the point of application of the force ($dr = -dx$). Therefore, in the given case the magnitude of $\frac{d|F|}{dx}$ is negative.

Likewise, the force of attraction of a magnet has a negative coefficient of recovery (but here the coefficient of recovery is not yet constant in absolute value), since it increases as the body being attracted approaches the magnet, i.e., as it moves in the direction of action of the force, which, by our terminology, is a negative displacement.

In certain individual instances, forces may exist with such small coefficients of recovery that they may be considered zero for all practical purposes. An example of this is terrestrial magnetism (which does not, of course, produce very large displacements of the attracted body). Strictly speaking, it has a slightly perceptible

negative coefficient of recovery.

If, at the point of application of a force F , having a positive coefficient of recovery, an additional force T of constant value is applied, acting opposite to the force F , then as soon as the force F (due to a displacement Δx of the point of application) reaches a magnitude T , we attain stable equilibrium in the system; thereafter, on positive displacement dx of the point of application of the forces, we obtain a force F , greater than the force T , resulting in a reverse displacement of the point of application. In the case of negative displacement dx , the force F becomes smaller after equilibrium has been attained than the force T , resulting in a displacement of the point of application in a positive direction.

If, however, the force F has a negative coefficient of recovery, then, given a position of the point of application of the forces in which the force F is equal to a constant force T acting at the same point in a direction opposite to the force F , the position of equilibrium of the system will naturally be unstable, since any negative displacement dx from the position of equilibrium will result in an increase in the force F , leading to a further displacement of the point at which the forces are displaced in the negative direction. Centrifugal force is an example of the above.

Thus, forces having positive coefficients of recovery may also be called forces acting on the principle of stable equilibrium, while forces having negative coefficients of recovery are forces acting on the principle of unstable equilibrium.

Total Coefficient of Recovery of a Number of Positional Forces in the Case of Constant Transmission Ratios Between Displacements of the Points at which Forces are Applied

In investigating the addition of the coefficients of recovery of a number of positional forces, we examine the simplest special case, i.e., that in which positional forces are applied to a single point in the mechanism, or in which, on movement of the mechanism of displacement $x_1, x_2, x_3 \dots$, the points of application of the po-

sitional forces $F_1, F_2, F_3 \dots$ are directly proportional to each other, i.e., when the transmission ratios $\frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}, \frac{dx_4}{dx_1} \dots$ are constant or vary so slightly as to be considered constant for all practical purposes.

We know that in these cases, with the established definitions, the coefficients of recovery of all the positional forces active in the mechanism are always superimposed; in this summation, the coefficient of recovery of each of the individual forces is always given the sign it has taken in isolation, regardless of the direction in which these forces tend to move the mechanism and regardless of whether they tend to move in opposite directions. We also know that, at the given constancy of transmission ratios, this rule remains in force even in the case of preliminary reduction of the coefficients of recovery to some single specific point in the mechanism.

In this case, when the positional force is reduced to some point or axis of the mechanism, the coefficient of recovery of the positional force has to be multiplied by the square of the ratio $\frac{dx_i}{dx_0}$ or $\frac{dx_1}{dx\phi}$, where dx_i is the simple displacement of the point of application of the force, dx_0 is the corresponding simple displacement of the point to which the forces are reduced, and $d\phi$ is the corresponding simple rotation of the axis of reduction of the forces. We may easily convince ourselves of this by means of elementary and familiar derivations.

However, if we have a combination of positional forces with positive coefficients of recovery and forces with negative coefficients of recovery, the result will be that the total of the absolute magnitudes of the reduced negative coefficients of recovery are subtracted from the total of the reduced positive coefficients of recovery. Therefore, in individual cases, we may emerge with a total coefficient of recovery equal to zero, regardless of the fact that, taken individually, each force at work in the mechanism has a coefficient of recovery of adequate magnitude. This is natural, since stable and unstable equilibria are contradictory concepts, and therefore, when taken in combination they compensate each other to

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some degree, this compensation being total in special cases.

By way of analogy we may conceive a balanced lever, able to rotate about the horizontal axis and mounted approximately vertically. We can divide this lever into two halves, of which one (in which the center of gravity is below the axis of rotation) tends to establish the lever in a position of stable equilibrium, while the other (with the center of gravity above the axis of rotation) tends to give the lever an unstable equilibrium specifically in the position in which the first half provides stable equilibrium. As a result, the combined action of the two halves of the lever yields a neutral equilibrium. The reaction of any positional forces whatever, where there are positive and negative coefficients of recovery, may be of the same nature.

By way of example let us consider a centrifugal friction governor reduced to its barest essentials. Let the entire system within the ring (3) in Fig.49 rotate

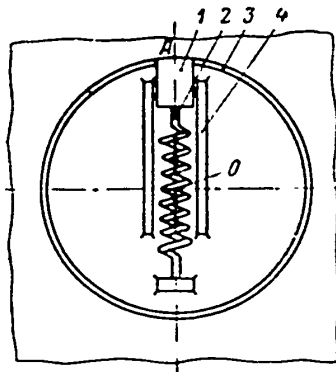


Fig.49 - Simplified Design of Centrifugal Friction Governor

about an axis 0, normal to the plane of the paper, while the ring (3) itself is fixed.

Let the weight (1) be displaceable relative to the system along the guides (4) i.e., in a straight line from the center 0. The weight is pulled toward the center 0 by the spring (2). As a centrifugal force is created on rotation of the system, the weight tends to move away from the center 0 and, resting on the ring (3), produces the desired braking of the mechanism

as a result of the generation of a friction force.

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We are describing here a somewhat simplified governor design, since in reality the weight, relative to the system, does not move along guides but by rotation about

an axis, so that in determining the conditions of equilibrium of the weight it is usually moments rather than the forces themselves that are involved, However, we are using a simplified design since our problem is to show only the principles involved. The introduction of the corresponding changes into the derivation, to obtain virtually the same result with the standard type of centrifugal governor, of course presents no particular difficulties.

In addition, the axis of rotation of the weight itself is usually so arranged that the arms of the moments acting in the mechanism of the forces undergo virtually no change when a weight is slightly displaced relative to the position in which it rests in the ring (3), since these arms ordinarily are almost normal to the line of action of the forces, when the mechanism is in the position shown. Therefore, the derivation developed below for a centrifugal governor of simplified design is almost completely applicable to a governor of the usual type.

It is common knowledge that, even in a case of stabilized rotation at constant velocity, this velocity of rotation of a mechanism provided with a governor of the type indicated will not always be identical. The fact is that even if we do not consider forces of inertia and damping in displacements of the weight along the guides (4), we will have the following equation for the forces acting upon the weight (1):

$$Sa + F - mr\omega^2 = 0, \quad (7.1)$$

where S is the elasticity of the spring;

a is the deflection of the spring;

m is the mass of the weight;

r is the distance from the center of rotation O to the center of gravity of the weight;

F is the force exerted by the weight against the ring (3);

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ω is the velocity of rotation of the governor.

Therefore,

$$\omega^2 = \frac{Sa + F}{mr}.$$

The distances r and a do not change in the given design. Thus, the velocity of rotation ω depends to some degree on the force that has to be exerted by the weight on the ring (3) to produce the required braking. This force F is equal to

$$F = \frac{M_p - M_c}{R\mu},$$

where M_p is the moment of the forces driving the mechanism (from some prime mover, for example) reduced to the governor axis;

M_c is the moment, reduced to the governor axis, required by the mechanism to maintain the speed. For which the given governor has been installed, this absorption of moment being due to the presence of friction and other resistances in the mechanism (the moment M_c does not include the moment absorbed by the governor itself);

R is the radius of the ring (3);

μ is the coefficient of friction between weight and ring.

Here, in the first place, the possibility of fluctuations in the moment M_p exists. However, the quantities most subject to variation are M_c and μ . Therefore, the required magnitude of the force F may undergo marked oscillations, which will be reflected in the magnitude of ω , i.e., in the accuracy of operation of the governor even from the viewpoint of its static errors alone.

In the above governor design we also provide for the force of the spring with its positive coefficient of recovery and for the centrifugal force with its negative coefficient of recovery, but the given design does not make use of the advantages of this combination. This is due to the fact that, when the governor is in operation, the weight rests in a rigid ring (3) and is unable to move elsewhere, with the re-

sult that the effect of the positional forces is lost.

This being the case, a governor of this type may be so designed that the contact between ring and weight is not rigid but elastic. Figure 50 presents a sample

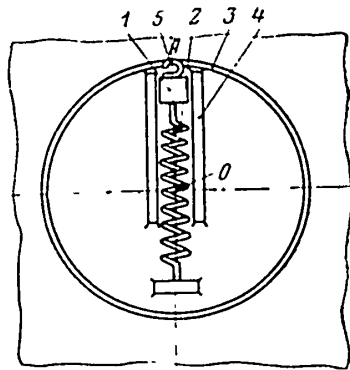


Fig.50 - Simplified Schematic for a
Centrifugal Friction Governor with
Additional Spring (5)

design for such a device. Attached to the weight (1), in addition to the spring (2), is another spring (5) which, by its pressure on the ring (3), creates the friction required for braking. The depiction of the spring (5) in Fig.50, is schematic, since in reality it obviously has to be rotated through 90° relative to the axis CA, as compared to its position in Fig.50. This is necessary so that the force of friction with the ring does not induce a further deflection of the

spring (5). Since the spring (5) is always in contact with the ring (3) when the governor is in operation, additional stops may be provided which do not permit the weight to deviate further to the center as soon as the spring (5) begins to be in contact with the ring (3).

In this design, the forces of the springs (2) and (5) act against the weight in the direction toward the center of rotation 0. The centrifugal force is operative in the opposite direction. Therefore, eq.(7.1) takes on the form

$$S_1 a + S_2 b - m r \omega^2 = 0, \quad (7.2)$$

where S_1 is the elasticity of the spring (2);

a is the deflection of the spring (2);

S_2 is the elasticity of the spring (5);

b is the deflection of the spring (5).

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The magnitude of S_2b is obvious, and constitutes the force F , exercised by the spring (5) against the ring (3) while the moment of friction M_{Tp} , created by the governor, is

$$M_{Tp} = S_2 b R \mu.$$

Let the magnitude of the required force F be changed (by varying the magnitudes M_p , M_c , and μ). Now it must equal $S_2 b_1$, while $b_1 - b = \delta$. However, this will be accompanied by changes in r and ω of the same magnitude δ . As a result, we will have the following equation:

$$S_1(a + \delta) + S_2(b + \delta) - m(r + \delta)\omega_1^2 = 0, \quad (7.3)$$

where ω_1 is the new rate of rotation.

If we now require that $\omega_1 = \omega$, and then find the difference between eqs. (7.2) and (7.3), we will obtain

$$(S_1 + S_2)\delta - m\omega^2\delta = 0,$$

or

$$S_1 + S_2 = m\omega^2. \quad (7.4)$$

Thus, it is required that the total coefficient of recovery $S_1 + S_2 - m\omega^2$ of the positional forces in such a system be equal to zero; therefore, when this condition is met we have an example of a case in which the coefficients of recovery of opposing sign compensate each other.

Having chosen the elasticities S_1 and S_2 of the springs (2) and (5) in such a fashion as to satisfy eq. (7.4), we obtain, in a case of a stabilized rate of rotation and stabilized position of the weight, one and the same velocity of rotation ^{STAT} of the controller, regardless of the magnitude of the force of contact F required. The slightest rise in the magnitude of ω will cause the centrifugal force to be greater

than the force of the springs, regardless of the position of the weight, and the latter will be strongly pressed against the ring (3), creating an intense braking action. The slightest reduction in ω causes the force F to decline sharply. As a result, we have a very sensitive governor.

A shortcoming of this design is the inconvenience of adjusting it at the required rate of rotation ω . In reality, we see from eq.(7.4) that a variation in the magnitude of ω while keeping condition (7.4) valid, is possible only by a corresponding change in the magnitudes of S_1 and S_2 . The magnitudes of a and b of the initial tensions on the spring do not even enter into this equation. If we merely vary the initial spring tension, then, despite the fact that a new magnitude of ω will appear to satisfy eq.(7.2), the condition (7.4) will be violated.

Thus, the adjustment of a device of this type may be performed only by varying the elasticity of the spring, which may require a corresponding variation in the magnitude of the initial tension thereon. Of course, all this is somewhat complicated in point of design. It is quite probable that, with more detailed elaboration, it will prove possible to arrive at a design that is more or less successful in point of adjustment, but in the given article we suggest only the idea of a governor along these lines and not its final design.

Another, more serious drawback of this design is the following: The derivations made above do not allow for the forces of inertia as the weight is displaced along the guides (4), while the weight has not yet been fixed in a position of equilibrium. However, it must always be remembered that one serious shortcoming of this type of governor with zero positional force is the fact that its centrifugal weight must always undergo fluctuations of considerable amplitude along the straight line OA.

A consideration of this factor may perhaps necessitate some reduction in tSTAT sensitivity of the governor and require a procedure in which the sum of $S_1 + S_2$ does not exactly equal the magnitude $m\omega^2$, but would perhaps be somewhat larger. In this

case, we would not obtain the same "ideal" effect as when eq.(7.4) is adhered to, but it is apparently possible nonetheless to follow the design in Fig.43 and build a governor, providing less variation in the velocity of ω relative to fluctuations in

the magnitudes of M_p , M_c , and μ than the governor built in accordance with the design of Fig.49.

In addition, it would also be possible to attempt the following: to provide some damping motion of the centrifugal weight in the direction of the straight line OA , and thereby to have the force resulting from a rotation of the system in the direction OA be due not only to the existence of the velocity of rotation of the governor ω (i.e., not only to centrifugal force), but also to the appearance

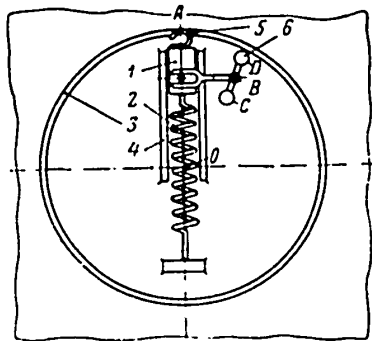


Fig. 51 - Simplified Schematic of a Centrifugal Friction Governor with Additional Control for Angular Velocity

of an acceleration ω of that velocity.

This may be reached, of course, by various types of design. For example, it would be possible to add a three-arm lever (6) able to rotate about the axis B (Fig.51). At the ends, C and D, of the arms BC and BD, weights p_1 and p_2 would be provided, so that $BCp_1 = BDp_2$. This last requirement is introduced so that no moment due to the centrifugal force can appear at the lever (6) relative to the axis B when the lever is in positions in which the points O, C, B, and D are not in a single straight line.

In such a design, the acceleration ω creates a moment relative to the axis B, due to the fact that the weight at the D end of the lever is farther from the center of rotation C than the weight at the C end of the lever. The part (6) is linked to the weight (1) in some manner, via its third arm. If eq.(7.4) is adhered to, the

equation of forces in the direction OA will have the form

$$M\ddot{x} + H\dot{x} = m(r_0 + x)(\omega^2 - \omega_0^2) + N\dot{\omega},$$

where M is the total reduced mass of the weight (1) and the part (6);

H is the damping coefficient;

ω_0 is the velocity of rotation at which eq. (7.4) is satisfied;

N is some dimensional magnitude characterizing the effect of the additional part (6) at angular accelerations of the governor, depending on the moment of inertia of the part (6);

the difference $\omega - \omega_0$ will, in the future, be denoted by $\Delta\omega$.

For small oscillations, in which we may, for purposes of simplicity, ignore those terms in the equation containing $(\Delta\omega)^2$, and also for terms containing the product $x\Delta\omega$ (considering these terms to be small by comparison with other terms of the equation), we have, approximately

$$M\ddot{x} - H\dot{x} \approx 2mr_0\omega_0\Delta\omega + N\dot{\omega}$$

On the other hand, we may write, roughly

$$\Delta\dot{\omega} = f(t) - kx,$$

where k is a constant;

$f(t)$ is the effect of the perturbation moment due to various fluctuations in the moment M_p of the drive, the moment M_c of the resistances in the mechanism, and the coefficient of friction μ .

From these two equations, we obtain

$$\text{and } \left. \begin{aligned} \Delta\ddot{\omega} + \frac{H}{M}\Delta\dot{\omega} + \frac{Nk}{M}\Delta\omega + \frac{2m}{M}kr_0\omega_0\Delta\omega &= f(t) + \frac{H}{M}\dot{f}(t) \\ \ddot{x} + \frac{H}{M}\dot{x} + \frac{Nk}{M}x + \frac{2m}{M}kr_0\omega_0x &= \frac{N}{M}\dot{f}(t) + \frac{2m}{M}r_0\omega_0f(t) \end{aligned} \right\} \quad (7.5) \text{ STAT}$$

Therefore, the equation for the natural vibrations of the system is

$$\ddot{x} + \frac{H}{M} \ddot{x} + \frac{Nk}{M} \dot{x} + \frac{2m}{M} kr_0 \omega_0 x = 0.$$

In accordance with Gurvits' criterion, the requirement for stability in the process is

$$\begin{vmatrix} H & 2mkr_0\omega_0 \\ M & Nk \end{vmatrix} > 0 \quad (7.6)$$

or

$$HN > M \cdot 2mr_0\omega_0$$

It is apparent that this last condition can be satisfied.

Among other things, it is also evident from eqs.(7.5) that if, after some variation in the moment of the drive or in the moment of resistance of the mechanism, these variations are retained and reach some constant magnitude, i.e., if $f(t)$ becomes equal to some constant C , then after the natural vibrations of the system have decayed, we will have $\Delta\omega = 0$ and $x = \frac{1}{k}C$, which is also derived from our prior conclusions.

It is also clear from the determinant (7.6) that, in the case of lack of damping and a design of the type of part (6) (i.e., when $H = 0$ and $r = 0$); the governor is unstable.

However, the design presented in Fig.51 is, of course, only one of the design variants generally possible for a governor of the type in question.

Similar changes may also be introduced in the design of an electric centrifugal governor. Figure 52 shows the general design of such a governor. Until the velocity of rotation ω of the governor about the axis O normal to the plane of the paper reaches a given magnitude, the centrifugal weight (1) is pressed against the contact (3) by the spring (2) with a particular force, and closes the electric cir-

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cuit feeding the drive of the mechanism. After a particular rate of rotation ω has been attained, the resultant centrifugal force reaches a magnitude at which the initial tension on the spring is overcome, the weight (1) is lifted off the contact, and the current feeding the mechanism is broken.

The fact that this type of governor suffers some irregularity, meaning that it

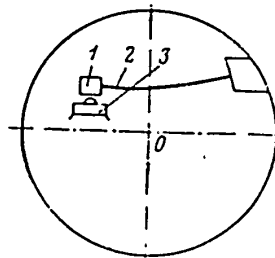


Fig.52 - Schematic of the Electric Centrifugal Governor

does not always sustain the same rate of rotation, is explained by the following causes: On the one hand, in order for the current to be broken, a given minimum gap is needed between the weight (1) and the contact (3). This, of course, requires a somewhat greater centrifugal force, and consequently a somewhat greater velocity of rotation of the governor than when the

weight is merely touching the contact. On the other hand, in order for the current to be turned off, a certain compressive force against the contact is needed, for which the rate of rotation of the governor must be somewhat slower than in the situation in which the weight merely touches the contact (3) without exerting pressure. Preliminary calculations for such a governor usually allow for the two considerations to which we have referred.

The inaccuracy of a governor due to these causes can be reduced only within given limits, by some reduction in the stiffness of the spring (2) and a corresponding increase in the degree of its initial deflection. However, this is not always possible, since excessive tensions in the spring are impermissible.

These causes of inaccuracy in governors of the type in question are explained by the fact that here we are dealing with the same shortcoming as in the governor shown in Fig.49. That is, when the centrifugal force rests on the contact (3), the force of the spring and the centrifugal force cease to be positional, so that a com-

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bination of the coefficients of recovery of these forces is to no avail.

This type of governor may be varied in a manner analogous to that described above for friction governors. The contact (3) (Fig.53) may be designed as a spring (part 3), with allowance for the fact that the spring does not work as such when the current is off. Therefore, in order for the total elasticity of the working springs

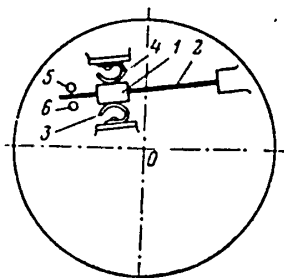


Fig.53 - Simplified Schematic of
Electric Centrifugal Governor with
Spring Contact (3) and Additional
Spring (4)

not to change during this time, it is necessary to add another spring (4), fixed to the housing of the governor. The weight (1) rests on this spring during the time when it is no longer in contact with the spring (3). The springs (3) and (4) have to be of identical elasticity and so mounted that when the weight (1) is in its middle position and is simultaneously in contact with both springs, they are under no tension. In other words, as soon as one of the springs begins to function, the

other ceases to be in contact with the weight.

Positions (5) and (6) in Fig.53 are stops to limit the motion of the weight, and do not affect the principle of operation of the governor.

Here we present an approximated and schematic derivation of the manner in which the elasticities of the springs (2), (3), and (4) should be determined. To wit, we will consider the total centrifugal force to be concentrated at a particular point of the weight or, in other words, we will ignore the centrifugal force which, on rotation of the governor, is created at points within the spring (2). If necessary, however, it is possible to make also calculations with allowance for these latter STAT forces. In this connection, it may be remarked that, in this governor, the spring (2) may be of a design other than that shown in Fig.53, where it is presented

only in schematic form.

In this design we have the following: When the weight is in its central position, i.e., when it only touches the springs (3) and (4) but does not yet press against them, we have the following force equation (applying, of course, to the steady state, in which there are no forces of inertia due to displacement of the weight normal to the spring 2):

$$S_1 a - m r \omega_0^2 = 0, \quad (7.7)$$

where S_1 is the elasticity of the spring (2);

a is the depth of curvature of the spring (2), with the weight in the given position,

r is the distance from the center of gravity of the weight to the center of rotation O , with the weight in the position indicated.

When the weight is in a position in which it presses against the spring (3), we have the following force equation

$$S_1 (1 - \delta_1) - S_2 \delta_1 - m (r - \delta_1) \omega_1^2 = 0, \quad (7.8)$$

in which S_2 is the elasticity of the spring (3);

δ_1 is the depth of curvature of this spring,

ω_1 is the rate of rotation of the governor at which the weight, in the given position, is in equilibrium.

In a case in which the weight is pressing against the spring (4) (the contact having been broken), we have the equation

$$S_1 (a + \delta_2) + S_2 \delta_2 - m (r + \delta_2) \omega_2^2 = 0, \quad (7.9)$$

where δ_2 is the depth of curvature of the spring (4);

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ω_2 is the rate of rotation of the governor, corresponding to the given position of the weight.

Let us consider the elasticity S_2 of the spring (4) as equal to that of the spring (3).

If we assume that $\omega_1 = \omega_0$, then eqs.(7.7) and (7.8) yield

$$-S_1\delta_1 - S_2\delta_1 + m\delta_1\omega_0^2 = 0,$$

from which we derive

$$S_1 + S_2 = m\omega_0^2 \quad (7.10)$$

regardless of the magnitude of δ_1 .

If, however, we make the condition that $\omega_2 = \omega_0$, then eqs.(7.7) and (7.9) give

$$S_1\delta_2 + S_2\delta_2 - m\delta_2\omega_0^2 = 0,$$

or again

$$S_1 + S_2 = m\omega^2.$$

Thus, when the elasticities of the springs (2), (3), and (4) are calculated so as to permit condition (7.10) to be satisfied, any rise in the rate of rotation of the governor to the slightest degree above the specified level, will make it impossible for the disturbed equilibrium of the system to be compensated by a corresponding variation in the spring tension. The result is an immediate large deflection of the spring (4) to the stop, thus resulting in an adequate gap between the contacts. The slightest decline in the rate of rotation immediately produces an adequate deflection of the spring (3), i.e., an adequate pressure on the contact.

The considerations advanced relative to the shortcomings of the type of friction governor depicted in Fig.50 are applicable to the governor now under discussion, which is of similar design. Also equally valid is the idea that, for practical purposes, it may be more convenient not to adhere too accurately to condition (7.10) STAT and that, in this connection the design given in Fig.53 may be more desirable than

that in Fig.52. In addition, it may be permissible here too, to add parts similar to those indicated for the friction governor, in order to secure stable operation of a governor.

Of course, the cases we have just described, in which the total coefficient of recovery of several positional forces at work in the mechanism may add up to zero, constitute relatively rare special cases. In the general case, however, we may only say that we will always emerge with a stable equilibrium for the mechanism in cases in which the total of the coefficients of recovery of positive positional forces, is greater than the absolute magnitude of the total of the coefficients of recovery of negative positional forces. If this were not the case, we would obtain an unstable equilibrium.

As an example of such a combination, let us examine the reaction between the force of a magnet and that of a spring. For example, let these forces act in opposing directions and furthermore, let some third force of constant magnitude operate at the point of application of these two forces. Let this be, say, the force of gravity, and assume even that its direction is the same as that in which the force of the spring is exerted. This will give us the following force equation:

$$F_{pr} + F_0 - F_M = 0,$$

where F_{pr} is the strength of the spring;

F_0 is the constant force;

F_M is the force of the magnet.

The relationship between the magnitudes of these forces and the points at which they are applied is shown in Fig.54, where the forces are plotted on the ordinate and the distance x , from the point of application of the forces to the origin, is plotted on the abscissa. The curve AB represents, approximately, the dependence of the magnetic force on the distance x . The straight line OC shows the dependence of the strength of the spring on the same distance. The straight line DE depicts the

magnitude of the constant force. The broken line K_1C_1 depicts the dependence of the total of the spring force and the constant force on the distance x . The broken line A_1B_1 is the mirror image (based on the Ox axis) of the curve AB .

The abscissas ON_1 and OK_1 of the points N and M of intersection between the

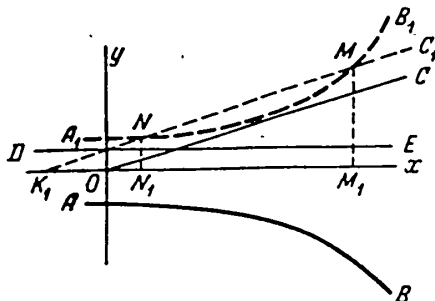


Fig. 54 - Graph of the Interaction of Magnet, Spring, and Constant Force

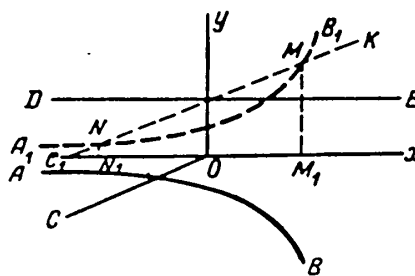


Fig. 55 - Graph of the Interaction of Magnet, Spring, and Constant Force

broken lines and the straight line correspond to the position of equilibrium of the point of application of these forces. In other words, in these positions the total of all the forces is equal to zero. It is obvious that the point N_1 corresponds to a stable position of equilibrium of the mechanism, since in this position the straight line K_1C_1 , characterizing the strength of the force with a positive coefficient of recovery, is steeper, relative to the abscissa, than the curve A_1B_1 describing the magnitude of the force with a negative coefficient of recovery. As far as the point M_1 is concerned, it corresponds to an unstable position of equilibrium of the mechanism, as here the force with a negative coefficient of recovery has a coefficient greater in absolute magnitude than the force with the positive coefficient.

An analogous picture is obtained in a case in which the force of the spring and that of the magnet act in the same direction and are opposed by a force of constant magnitude, or in other words when we have the equation

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$$F_0 - F_u - F_{pr} = 0.$$

This case is depicted graphically in Fig.55, with the same symbols used in Fig.54. Here the point M_1 also corresponds to a stable position of equilibrium in the system, and the point M_2 to a position of unstable equilibrium.

General Case of Reduction of Coefficients of Recovery of Positional Forces

Let us examine the general case of a reduction in the coefficients of recovery of positional forces. In other words, let us now also take into consideration the fact that the transmission ratio between the displacement of the point at which a force is applied (or the axis of application of a moment) and the displacement of the point (or axis) of reduction of the force may be variable. In this general case, it may not only prove that the functional dependence of the reduced positional force and its coefficient of recovery due to displacement of the point of application of the force is completely unlike the functional dependence of the force not yet reduced (and the coefficient of recovery of the latter) upon the displacement of the point of application of the force, but, as indicated below, cases are also entirely possible in which the coefficient of recovery may, on reduction, even change its sign or be reduced to zero. Simple examples of such cases can be derived even prior to the derivation of the general formula for the reduction coefficients of positional forces.

Let us take the following case as an example: Let the force acting in the mechanism create a moment relative to a given axis, and, on simple displacement of the mechanism under the effect of this force, let the arm of the given moment also change in magnitude, the relative change in this arm being of the same order as the relative change in the magnitude of the force. In this situation, it may happen that the lessening of the force is accompanied by a diminution in the arm for the moment of force, or that it may increase with a reduction in the force. In the lat-

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ter case, the relative change in the size of the arm may prove to be even larger than the relative change in the force itself.

Thus, cases are entirely within the realm of possibility in which displacements of the mechanism with increase in the magnitude of the force are accompanied by a decline in the moment thereof, and others in which the moment increases although the force decreases. Therefore, if such a force is reduced to some further point in the mechanism, it is quite possible that the reduced force will decline with an increase in the force applied and will increase as the latter declines. However, the result of this is that, when the mechanism moves in the direction in which the force acts, the applied force is diminished and the reduced force increased or, conversely, the applied force increases and the reduced force declines, i.e., the coefficients of recovery of the applied and reduced forces are opposite in sign.

This phenomenon is possible not only when the applied force creates a moment whose arm is variable in magnitude, but also generally in cases in which the transmission ratio of the displacement of the point of application to the point of reduction of the force is not constant.

All this is readily confirmed by means of simple analytical derivations. We know that the expression for the reduced force F_{pr} has the form

$$F_{pr} = |T| \frac{dy}{dx}, \quad (7.11)$$

where T is the force applied, while dy is the simple displacement of the point of application of the force (which, as before, we will consider positive in a direction opposite to that in which the force is exerted) corresponding to the simple displacement dx of the point to which the force is reduced in the given direction.

Thus, if, on positive displacement of dx , the displacement dy is also positive, i.e., if the force tends to shift the system toward the negative displacement dx , the reduced force F_{pr} is positive, this also being the result obtained from eq. (7.11).

When the direction of the force T is opposite to this, the reduced force F_{pr} is negative. Thus, we must always use the absolute value of the force itself in eq.(7.11).

Let us suppose that, at a point in the mechanism to which the force is reduced, the final result is some algebraic sum P of the reduced forces

$$P = F'_{pr} + F''_{pr} + F'''_{pr} + \dots$$

and let us suppose that the point to which all the forces have been reduced is displaced by a magnitude dx . Then the total force P changes by a magnitude dP , equal to

$$dP = dF'_{pr} + dF''_{pr} + dF'''_{pr} + \dots = d|T_1| \frac{dy_1}{dx} + d|T_2| \frac{dy_2}{dx} + \\ + d|T_3| \frac{dy_3}{dx} + \dots + |T_1| d\left(\frac{dy_1}{dx}\right) + |T_2| d\left(\frac{dy_2}{dx}\right) + |T_3| d\left(\frac{dy_3}{dx}\right) + \dots$$

In this case, the points of application of the forces are displaced correspondingly by $dy_1, dy_2, dy_3, \dots$. In accordance with the symbols adopted (i.e., considering the concepts we have accepted as to the positive and negative coefficients of recovery of positional forces) we may write, in this case

$$dP = S dy,$$

where S is the coefficient of recovery of the force (positive or negative). Therefore, we have

$$S_{tot} dx = S_1 dy_1 \left(\frac{dy_1}{dx}\right) + S_2 dy_2 \left(\frac{dy_2}{dx}\right) + S_3 dy_3 \left(\frac{dy_3}{dx}\right) + \dots \\ + |T_1| \frac{d^2 y_1}{dx^2} dx + |T_2| \frac{d^2 y_2}{dx^2} dx + |T_3| \frac{d^2 y_3}{dx^2} dx + \dots$$

where S_{tot} is the total reduced coefficient of recovery of the overall positional force of the system,

$S_1, S_2, S_3, \dots$ are, respectively, the coefficients of recovery of the forces $T_1, T_2, T_3, \dots$

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From this we have

$$S_{tot} = S_1 \left(\frac{dy_1}{dx} \right)^2 + S_2 \left(\frac{dy_2}{dx} \right)^2 + S_3 \left(\frac{dy_3}{dx} \right)^2 + \dots$$

$$\dots + |T_1| \frac{d^2 y_1}{dx^2} + |T_2| \frac{d^2 y_2}{dx^2} + |T_3| \frac{d^2 y_3}{dx^2} + \dots \quad (7.12)$$

Thus, in a case in which the $\frac{dy}{dx}$ ratio is constant and $\frac{d^2 y}{dx^2}$ is zero, an expression in the form $S \left(\frac{dy}{dx} \right)^2$ will give the reduced coefficient of recovery of one or another force acting in the mechanism. Since, in this case, $\frac{dy}{dx}$ is squared and the factor of reduction is consequently positive, regardless of the sign of the $\frac{dy}{dx}$ ratio itself, we find that the coefficients of reduction in this case are always entered into the summation with the same signs they had before reduction. In this connection, it is immaterial whether the forces seek to displace the mechanism in one direction or in opposing directions.

However, if the expression $\frac{dy}{dx}$ is not constant, then on the basis of eqs.(7.12) we have

$$dF_{pr} = S dy \left(\frac{dy}{dx} \right) + |T| \frac{d^2 y}{dx^2} dx =$$

$$= |T| \frac{dy}{dx} \left[\frac{dT}{|T|} + \frac{d \left(\frac{dy}{dx} \right)}{\frac{dy}{dx}} \right], \quad (7.13)$$

from which we find that the reduced coefficient of recovery S_{pr} of the force F_{pr} is equal to

$$S_{pr} = S \left(\frac{dy}{dx} \right)^2 + |T| \frac{d^2 y}{dx^2} = S \left[\left(\frac{dy}{dx} \right)^2 + \frac{|T| \frac{d^2 y}{dx^2}}{S} \right]. \quad (7.14) \quad \text{STAT}$$

The following is evident from the derived equations: If, the relative increase in the absolute magnitude of the force T is followed by a relative reduction in the

magnitude of $\frac{dy}{dx}$, i.e., if the quantity $\frac{d\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}}$ has a sign opposite to that of the quantity $\frac{d|T|}{|T|}$ and if the relative variation in the magnitude of $\frac{dy}{dx}$ is greater than the relative change in the force T , then the quantities dF_{pr} and $d|T|\frac{dy}{dx}$; and the quantities S_{pr} and S are of different sign.

In addition, we see the following from these equations: If the force T has a constant magnitude, regardless of the displacements of the mechanism (i.e., if the force we are dealing with is not positional), while the expression $\frac{dy}{dx}$ changes in magnitude on any displacement of the mechanism, then the magnitude of the reduced force F_{pr} is dependent upon the position of the mechanism, i.e., the reduced force acquires the properties of a positional force.

The simplest example of this phenomenon is the ordinary pendulum. It is true that while the coefficient of recovery of the force of gravity itself is virtually zero, the magnitude of the moment of this force is nonetheless a function of the angle by which this pendulum deviates from a position of equilibrium. Consequently, this moment has properties analogous to the properties of the moment of a compensated pendulum connected to a spring.

This is fully in agreement with eq.(7.14). The following is the actual consequence thereof: If $S = 0$, then

$$c_{pr} = |T| \frac{d^2y}{d\varphi^2},$$

where c_{pr} is the coefficient of recovery of the moment reduced to the axis of the pendulum,

φ is the angle of deviation of the pendulum.

In the given case, $T = Q$ (the weight of the pendulum). Further, displacement of the weight along the vertical equals

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$$y = l(1 - \cos \varphi),$$

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where l is the distance from the center of gravity of the pendulum to its axis of rotation. Since

$$\frac{dy}{d\varphi} = l \sin \varphi; \quad \frac{d^2y}{d\varphi^2} = l \cos \varphi,$$

then

$$c_{pr} = Ql \cos \varphi,$$

and this is in full agreement with the case of a physical pendulum differing from the vertical by an angle φ .

Thus, the case in which the reduced force or the reduced moment proves to be positional, and the active reduced force itself has no coefficient of recovery, is common. It may therefore be suggested that the opposite case is also possible, i.e., one in which the reduced force (or moment) has no coefficient of recovery, although the active reduced force is itself positional (i.e., is the force of some spring). Below we will derive examples of such a case.

Likewise, in view of the fact that the coefficients of recovery of positional forces and moments may change in sign when they are reduced, the following case is also entirely within the realm of possibility. A mechanism may contain several positional forces, themselves presenting coefficients of recovery of a single sign (i.e., there may be several springs in a mechanism), but after reduction to a single point of the mechanism the reduced coefficients of recovery may prove to be of differing sign. As a result of the reaction of several springs, each of adequate elasticity, it is thus possible in certain cases to end up with a system of very small or even zero coefficient of recovery.

This last result would seem somewhat to contradict the generally accepted rule in accordance with which the reduced stiffnesses of a number of springs, operating in the same mechanism, are always totaled; therefore, the addition of any additional spring always increases the elasticity of the system, regardless of the direction in

which the individual springs are operative.

Here it must be noted, however, that this generally accepted rule is, in many cases, wholly applicable only since the springs in mechanisms are usually so mounted that the magnitudes of the transmission ratios $\frac{dy}{dx}$ (see definitions above) undergo very little variation on displacement of the mechanism, relative to the variation in the magnitudes of the forces set up by the springs themselves. These $\frac{dy}{dx}$ ratios are often virtually constant in value. In particular, springs are, as a general rule, never connected to levers in the mechanism in such a fashion as to establish a very acute angle between the line of action of the spring and the lever arm to which the spring is connected (the angles in question usually are nearly right angles). This is the reason for the fact that, if several springs are operative in a single mechanism, the reduced elasticities of these springs are always totaled.

In general it may be taken, as a matter of theory, that the following results are attainable by proper selection of the transmission mechanism. The positional force (or moment) constituting any given function $f(y)$ of the position y , at which it is applied, reduced to any other point (or axis) of the mechanism, may prove to be any other function $F(x)$ of the position x of the point (or axis) of reduction. In this connection, the function in accordance with which the transmission mechanism must be built, is easily demonstrated.

It is obvious that, in a simple displacement of the mechanism, the simple work done by the reduced force must be equal to the simple work of the driven force, i.e.,

$$F(x)dx = f(y)dy,$$

from which is obtained

$$\int F(x)dx - \int f(y)dy = c, \quad (7.15) \quad \text{STAT}$$

where c is some constant.

Let us derive specific examples of the interaction of positional forces, in

which the transmission ratios between the displacements of the points of application are variable.

For example, let it be required that the reduced force of some spring in a mechanism be rendered constant, i.e., that it cease to be positional after reduction. Let us take a simple spring of constant stiffness S ; then,

$$j(y) = Sy.$$

It is required that

$$F(x) = T,$$

where T is a constant.

In accordance with eq.(7.15)

$$\int T dx - S \int y dy = c;$$

$$Tx - c = \frac{Sy^2}{2},$$

or

$$T(x-a) = \frac{S}{2} y^2,$$

where

$$a = \frac{c}{T}.$$

By changing the point of origin from which the magnitudes of x are read, and, specifically, taking the reading x_1 from some new point of origin to be $x_1 = x - a$, we have

$$Tx_1 = \frac{S}{2} y^2. \quad (7.16) \quad \text{STAT}$$

If the functional ratio between the displacement x of the point to which the force is reduced and the displacement at the point of application of the force is

realized in accordance with this equation, a spring of variable force S_y will always be balanced by the same constant force in all positions of the mechanism, i.e., regardless of the force of tension on the spring itself.

It is very easy to plot a transmission satisfying eq.(7.16). We know that

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}. \quad (7.17)$$

Therefore, it is possible to plot a transmission of the following type: When a projection of the displacement of some point in the mechanism (to which we reduce the force of the spring) in the line of action of a constant force is equal to a magnitude proportional to $(1 - \cos \alpha)$, the spring acting in the mechanism must be stretched by a factor proportional to $\sin \frac{\alpha}{2}$. It is possible to adduce a number of typical designs of mechanisms satisfying this requirement.

Let us take, for example, the mechanism depicted in Fig.56. The platform AA_1 is

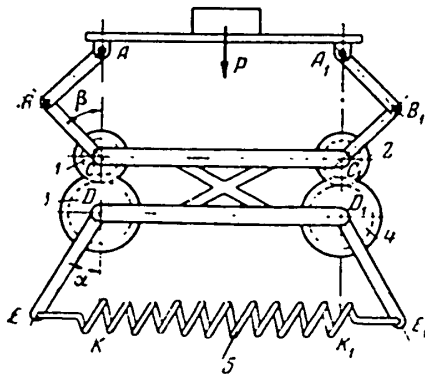


Fig.56 - Simplified Mechanism in Which the Reduced Positional Force is Constant (i.e., Ceases to be Positional)

hinged to the levers AB and A_1B_1 which, in turn, are hinged to the levers BC and B_1C_1 . Thus, $AB = A_1B_1 = BC = B_1C_1$.

The levers BC and B_1C_1 are rigidly fixed to the gear wheels (1) and (2). These latter mesh respectively, with the gear wheels (3) and (4), to which the levers DE and D_1E_1 are rigidly fixed (while $DE = D_1E_1$). The transmission ratio of the gears (1) and (3), and also of the gears (2) and (4), is $1/2$ so that the angle α of rotation of the levers DE and D_1E_1 is one-half the angle of rotation β of the levers CB and C_1B_1 . A spring (5) is fastened to the ends, E and E_1 , of the

levers DE and D_1E_1 . The length of this spring, in the free state, is DD_1 . Its flexure is $ED = E_1K_1 = 2EK$. A force P of constant magnitude, such as the force of gravity of a weight, is applied to the middle of the platform AA_1 .

In this mechanism, the displacement x of the weight P, along the vertical, is

$$x = 2AB(1 - \cos \beta) = 2AB(1 - \cos 2\alpha) \quad (7.18)$$

The flexure of the spring γ is

$$\gamma = 2DE \sin \alpha \quad (7.19)$$

Thus, the given mechanism satisfies condition (7.17), so that at one and the same constant weight P, equilibrium will result at any position of the mechanism. To verify this, we determine the force P and the force of the spring to the point of contact of the teeth of the gear wheels (1) and (3) and of the gear wheels (2) and (4).

In so doing we ignore the weight of the levers constituting the system. This is permissible since all the levers are readily balanced relative to their axes of rotation, and furthermore, it is possible to allow for the weights of the platform AA_1 , the levers AB and A_1B_1 , and their counterweights, adding this weight to the weight of the load P.

The forces P_1 , each equal to

$$P_1 = \frac{P}{2 \cos \beta}$$

act along the levers AB and A_1B_1 .

These forces create moments M_1 relative to the axes C and C_1 . Each of these moments has a value of

$$M_1 = \frac{P}{2 \cos \beta} CB \sin 2\beta = PCB \sin \beta$$

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(since the angle ABC is equal to 2β).

Thus, the force P, reduced to the point of contact of the gear wheels (1) and (3) [or of the gear wheels (2) and (4)], is equal to

$$Q_1 = \frac{PCB \sin \beta}{r \cos \delta},$$

where Q_1 is the reduced force;

r is the radius of the addendum line of the gear wheel (1);

δ is the angle of contact of the toothed wheels (1) and (3).

The force of tension F of the spring (5) is equal to

$$F = S2EK = 2SDE \sin \alpha,$$

where S is the stiffness of the spring.

The moments M_2 , created by this force relative to the axes D and D_1 , are equal to

$$M_2 = 2S DE^2 \sin \alpha \cos \alpha = S DE^2 \sin \beta.$$

Therefore, the force F, reduced to the point of contact of the gears (1) and (3) is equal to

$$Q_2 = \frac{SDE^2 \sin \beta}{2r \cos \delta},$$

where Q_2 is the reduced force.

The forces Q_1 and Q_2 act in opposite directions. The difference between them is

$$Q_1 - Q_2 = \left(PCB - \frac{SDE^2}{2} \right) \frac{\sin \beta}{r \cos \delta}.$$

Should PCB and $\frac{SDE^2}{2}$ be equal, the difference $Q_1 - Q_2$ always equals zero, regardless of the magnitude of the angle β , i.e., of the force F of tension on the

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spring.

However, if the force P is even a little larger than $\frac{SDE^2}{2CB}$, the platform AA_1 immediately moves downward until it rests on its support, since the forces Q_1 and Q_2 are incapable of balancing each other at any position thereof. However, if the force is just slightly less than the magnitude of $\frac{SDE^2}{2CB}$, the spring (5) immediately raises the platform to a position in which the levers AB and CB constitute a single straight line.

Likewise, it may be shown, on the basis of eq.(7.14) that the stiffness of the spring (5), reduced to the platform AA_1 , is equal to zero. In fact, if we exclude the angle α from eqs.(7.18) and (7.19), we obtain

$$\begin{aligned}x &= 4AB \frac{y^2}{4DE^2}; \\y &= DE \sqrt{\frac{x}{AB}}; \\\frac{dy}{dx} &= \frac{1}{2} \frac{DE}{\sqrt{ABx}}; \\\frac{d^2y}{dx^2} &= -\frac{1}{4} \frac{DE}{\sqrt{ABx^3}}.\end{aligned}$$

from which it follows that

$$S_{np} = S \left[\frac{1}{4} \frac{DE^2}{ABx} - \frac{SDE \sqrt{\frac{x}{AB}}}{S} \frac{1}{4} \frac{DE}{\sqrt{ABx^3}} \right] = 0.$$

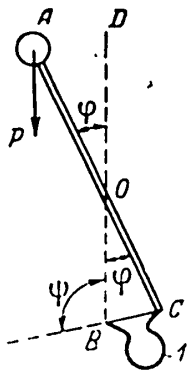
As another example of a case in which displacement, in a given direction, of the point of application of a constant force in a mechanism is proportional to $1 - \cos 2\alpha$, and the resulting extension of a spring in the mechanism is proportional to $\sin \alpha$, we may derive the following design (Fig.57).

Let a straight lever AC rotate about a horizontal axis O . The end A of the lever carries a weight P and the end C of the lever is provided with a spring (1), whose other end is mounted to the point B resting on a vertical, passing through the

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axis O. The distance OB is equal to the length of the OC arm of the lever. Let the flexure of the spring (1) be equal to the distance BC, i.e., let the ends of the spring meet in the free state.

Here, too, the displacement x in the vertical direction, of the load P is equal to $OA(1 - \cos \varphi)$ and the flexure of the spring is equal to $2 OC \sin \frac{\varphi}{2}$. Therefore here too we obtain, after the corresponding transformations,



$$S_{np} = S \left(\frac{dy}{dx} \right)^2 + S_y \frac{d^2 y}{dx^2} = 0.$$

When the lever deviates by an angle φ from the vertical DB, the total moment M produced by the action of the force of gravity and the effect of the force exerted by the spring (1) will be

Fig.57 - Another Simplified Mechanism in which the Reduced Positional Force is a Constant

$$M = POA \sin \varphi - SBC \sin \psi,$$

where S is the stiffness of the spring (1).

However,

$$BC = 2OB \sin \frac{\varphi}{2}; \quad \sin \psi = \cos \frac{\varphi}{2}.$$

From this, we derive

$$M = POA \sin \varphi - SOB^2 \sin \frac{\varphi}{2} OB \cos \frac{\varphi}{2} = (POA - SOB^2) \sin \varphi$$

Thus, if $POA = SOB^2$, we obtain a system in which the coefficient of recovery is zero, i.e., the active moments cancel each other at all angles φ or, in other words, regardless of the spring tension. However, if the force of gravity P is not equal to $\frac{SOB^2}{OA}$, then the two moments in question cannot compensate each other at any position of the lever. If the force is somewhat smaller than the magnitude of $\frac{SOB^2}{OA}$, STAT

the lever immediately occupies a vertical position in which the point C coincides with the point B, and if the force P is somewhat larger than the magnitude of $\frac{SOB^2}{OA}$, the lever deviates from the vertical to an angle $\varphi = \pi$.

Let us, further, examine a case of another transmission ratio between the motions of the parts of the mechanism. Let it be required now that the positional force of a simple spring of some constant stiffness S_1 , reduced to another point in the mechanism, has some constant negative stiffness $-S_2$ or, in other words, let it be required that two simple springs acting in a single mechanism and each independently having certain constant positive stiffnesses S_1 and S_2 , counterbalance each other regardless of the positions of the mechanism.

In this case, in accordance with eq.(7.15), we have

$$S_1 \int x dx + S_2 \int y dy = c,$$

or

$$S_1 x^2 + S_2 y^2 = c. \quad (7.20)$$

Apparently it is possible to develop a substantial number of designs of transmission mechanisms satisfying eq.(7.20), since the principle that the sum of the squares of two magnitudes is always equal to some constant is encountered with ample frequency in elementary geometry and trigonometry.

For example, if, in some mechanism, we have $x = OA \sin \varphi$ and $y = OB \cos \varphi$ and if, in this connection, the parameters of the mechanism are set so that $S_1 OA^2 = S_2 OB^2$, we derive

$$S_1 x^2 + S_2 y^2 = S_1 OA^2 \sin^2 \varphi + S_2 OB^2 \cos^2 \varphi = S_1 OA^2,$$

or, in other words, eq.(7.20) is satisfied.

It is therefore possible to use the following design:

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Let us take the articulated lever AOB (Fig.58), rotating about the axis O. The

angle AOB is $\frac{\pi}{2}$. The springs (1) and (2), creating the forces F_1 and F_2 acting parallel to each other, are attached to the ends A and B of the lever. If we drop the perpendicular CD, normal to the line of action of the springs (1) and (2), the flexure of the spring (1) is equal to the line AC, and that of the spring (2) is equal to

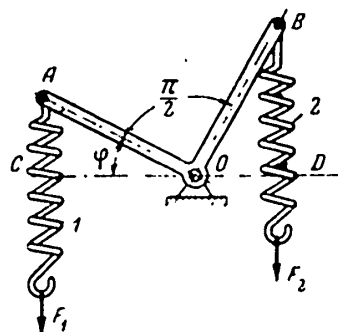


Fig.58 - Schematic Mechanism in which the Forces of Two Springs Compensate Each Other, Regardless of the Position of the Lever AOB

the line BD. In addition, let us suppose that the lines of action of the forces F_1 and F_2 remain parallel to themselves when the lever AOB is in various positions. This is approximately the case, for example, if the springs (1) and (2) are exceptionally long relative to the length of the lever arms OA and OB.

In addition, an accurate fulfillment of this condition may even be attained by the use of various simple added fittings. While some such fittings require that the deflection of the springs (1) and (2) is a specific number of times (let us say, twice) as large as that indicated above, this obviously does not in any way affect the results of the derivations made below.

In such a design, the total moment M , created by the forces of the two springs relative to the axis C, is

$$M = S_1 AC \cdot OC - S_2 BD \cdot OD = S_1 OA^2 \sin \varphi \cos \varphi - S_2 OB^2 \sin \left(\frac{\pi}{2} - \varphi \right) \cos \left(\frac{\pi}{2} - \varphi \right) = (S_1 OA^2 - S_2 OB^2) \sin \varphi \cos \varphi$$

Thus, in this case we find the difference between the absolute magnitudes of STAT the reduced stiffnesses of the two springs, while in a case in which $S_1 OA^2 = S_2 OB^2$, the forces of the springs (1) and (2) together create no moment relative to the O ax-

is of the lever in any position. In other words, the system is of zero elasticity, although the elasticities of the springs (1) and (2), acting on the lever, may each individually be adequate. Therefore, the slightest additional moment (over the moments of the springs) may theoretically (if the forces of friction are neglected) rotate the lever right to the limit, or in other words until one of the springs is taken out of action.

We will obtain the same result, of course, if we reduce the elasticities of both springs (1) and (2) by the same formula to a single point, say to the axis O. In this case, we will derive the following value c_{tot} for the total elasticity of the system

$$c_{tot} = S_1 \left[\left(\frac{dAC}{d\varphi} \right)^2 + \frac{|F_1| \frac{d^2 AC}{d\varphi^2}}{S_1} \right] + S_2 \left[\left(\frac{dBD}{d\varphi} \right)^2 + \frac{|F_2| \frac{d^2 BD}{d\varphi^2}}{S_2} \right].$$

We have

$$\begin{aligned} AC &= OA \sin \varphi; \\ BD &= OB \cos \varphi. \end{aligned}$$

From this, we obtain

$$\begin{aligned} \frac{dAC}{d\varphi} &= OA \cos \varphi; \\ \frac{d^2 AC}{d\varphi^2} &= -OA \sin \varphi; \\ \frac{dBD}{d\varphi} &= -OB \sin \varphi; \\ \frac{d^2 BD}{d\varphi^2} &= -OB \cos \varphi; \end{aligned}$$

and, moreover,

$$\begin{aligned} \frac{|F_1|}{S_1} &= OA \sin \varphi; \\ \frac{|F_2|}{S_2} &= OB \sin \left(\frac{\pi}{2} - \varphi \right) = OB \cos \varphi. \end{aligned}$$

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As a result we have

$$c_{tot} = S_1 [OA^2 \cos^2 \varphi - OA^2 \sin^2 \varphi] + S_2 [OB^2 \sin^2 \varphi - OB^2 \cos^2 \varphi] = \\ = (S_1 OA^2 - S_2 OB^2) (\cos^2 \varphi - \sin^2 \varphi),$$

i.e., we again arrive at the result we had previously attained, to wit, that the absolute magnitude of the reduced elasticity of one of the springs (1) and (2) is subtracted from the other.

From the last equation we also see that the general expression of the elasticity c_{pr} of either F_1 or F_2 , reduced to the axis O, takes the form

$$c_{pr} = SR^2 (\sin^2 \psi - \cos^2 \psi),$$

where S is the elasticity of the force under examination;

R is the arm of the lever corresponding to this force,

ψ is the absolute magnitude of the angle between this arm and the line of action of the force (i.e., for the force F_1 , an angle $\psi = \frac{\pi}{2} - \varphi$; and for the force F_2 , an angle $\psi = \varphi$).

From this we obtain the result that, for each of the two springs taken individually, the elasticity, reduced to the axis, is positive until $\sin^2 \psi - \cos^2 \psi > 0$, i.e., until the angle ψ is greater than 45° , and the reduced elasticity is negative, if the angle ψ is less than 45° .

Another example of the usefulness of eq.(7.20) is provided by a characteristic of parallelograms (and in special cases, of a rhombus) which is that the sum of the squares of its diagonals is always equal to the same constant, regardless of the variation in the angles between the sides of the parallelogram. This property has been used in the following simple design, to obtain a negative reduced elasticity of constant magnitude from the ordinary spring mounted in the mechanism.

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About two fixed axes A and A_1 (Fig.59), the levers AB and A_1B_1 are free to rotate ($AB = A_1B_1$). These levers are hinged, at the points B and B_1 , to the levers BC

and B_1C_1 of the same lengths as the former. These levers are hinged, at the points C and C_1 to the platform CC_1 ($CC_1 = AA_1$). For reasons of stability (to prevent the

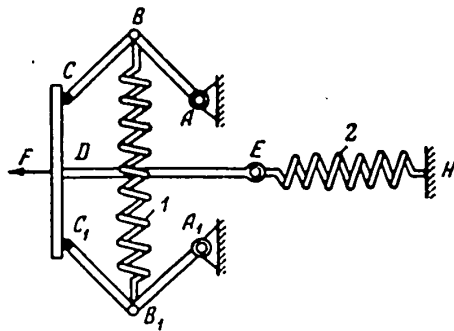


Fig. 59 - Mechanism in which the Stiffness of the Spring (1), Reduced to the Platform CC_1 , is Negative

levers from being displaced in a direction normal to the plane of the paper), the axes A, A_1, B, B_1, C , and C_1 can be made as long as necessary (in the direction normal to the plane of the paper). The axes B and B_1 are pulled together by the spring (1), whose tension is $BB_1 - AA_1$. Thus, the initial length of the spring outside the mechanism is $AA_1 = CC_1$.

Let us express the force F of the spring (1) reduced to the platform CC_1 , as a function of the value x of displacement of the platform in a direction opposite to the line of action of the force F . The tensile stress T of the spring (1) is equivalent to

$$T = S(BB_1 - AA_1) = 2Sl \sin \alpha,$$

where S is the elasticity of the spring;

l is the length of the levers AB, A_1B_1, BC or B_1C_1 ;

α is the angle $\angle CAB$.

This stress produces a force Q along the levers in question

$$Q = \frac{T}{2 \sin \alpha}.$$

As a result, the levers BC and B_1C_1 transmit to the platform CC_1 a stress F equal to

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$$F = 2Q \cos \alpha = 2Sl \cos \alpha.$$

Using x to denote the distance of the platform CC_1 from the line AA_1 , and taking the line opposite to the line AC as positive, we have, for this position of the mechanism

$$-x = 2l \cos \alpha,$$

from which it follows that

$$F = -Sx.$$

Thus, the reduced force F has a negative elasticity, i.e., it increases as a result of the motion of the mechanism in the direction of the force, while the elasticity is a positive value. Thus, the reduced force F may be balanced in any position of the mechanism (specifically, of course, in the range of motion of the platform CC_1 from $x = 0$ to $-x = 2l$) with the aid of an ordinary spring (2) of constant positive elasticity S [equal to the elasticity of the spring (1)], acting on the platform through the link ED and fastened at its other end to a fixed point H . For purposes of equilibrium of the system, the deflection of this spring (2) must equal the distance AC . In other words, in the position of the mechanism in which the platform CC_1 coincides with the straight line AA_1 , the tension on the spring (2) must be zero.

This design has the following characteristic: If we move the point H in any direction along the line MD , to any distance d , from a position in which the mechanism is in equilibrium, then a single force Sd will always be active at the point D , regardless of the degree to which the platform later shifts toward the spring (2) or in the opposite direction, i.e., regardless of the degree to which the magnitude of the tensile stress of the spring (2) itself is affected by this movement of the platform. Actually, the total effect of the forces F and P , after displacement of the point H , equals

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$$F + P = -Sx + S(x + d) = Sd,$$

i.e., it is not dependent on the magnitude of x .

Thus, this type of system may be used, for example, in a stand for testing auto-pilot directional gyro, the stand serving to imitate the behavior of an aircraft with a directional gyro, as in the Oppelt test stand. What is required in these stands is that deviations from the mid-position of the servo mechanism create moments relative to the axis of the stand proportional to the degree of deviation of the servo mechanism, but not dependent on the rotation of the axis of the stand itself. Such a rotation must not create moments of any kind.

In addition, mechanical designs of the types presented in Figs. 56 and 59, and other, analogous, schemes may be used in various mechanisms where the requirement is that the coefficient of recovery of a force developing due to tension on springs of some type be zero, or in other words that a force of identical magnitude always be attained regardless of changes in the spring tension.

The characteristic of parallelograms referred to above and used in Fig. 59 may be used in another way as well, to wit, if the sides of the parallelogram are rendered variable in length, while the lengths of the two diagonals are kept constant, the sum of the squares of the two adjacent sides will be a constant. In the special case in which the two diagonals are equal in length, the characteristic of the sides of the parallelogram, to which we have just referred, is also a characteristic of the legs of a right triangle, if its hypotenuse is constant in size.

By way of illustration, let us take the design shown in Fig. 60. At the C end of the lever OC, rotating about the axis O, two springs (1) and (2) are attached, whose other ends are mounted to fixed points A and B ($AO = OB$ and the points A, O, and B lie in a single straight line). In such an arrangement, we have

$$AC^2 + BC^2 = 2OA^2 + 2OC^2 = \text{const.}$$

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Therefore, to satisfy eq. (7.20) it is essential that the springs (1) and (2) are what is known as "null" springs, i.e., springs which, when wound, are given an

initial compressive stress between adjacent coils which would cause the ends of the spring to meet at a single point if the coils of the spring did not rest on each other. In other words, the tensile stress on such a spring is S/l where S is the elasticity of the spring and l its length in the given state.

If, in addition thereto, the elasticities of the springs (1) and (2) are equal,

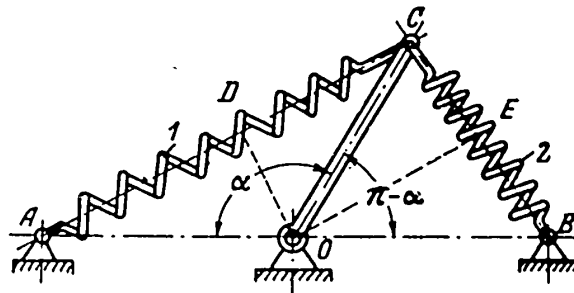


Fig.60 - Schematic Mechanism with Two Springs Compensating Each Other at all Positions of the Lever OC

then, in accordance with condition (7.20), the lever OC must be in equilibrium whatever its position.

Actually, the tensile stress of one spring is $S \cdot AC$, and that of the other is $S \cdot BC$. The moments of these stresses relative to the O axis are, correspondingly, $S \cdot AC \cdot OD$ and $S \cdot BC \cdot OE$. However, the product $AC \cdot OD$ is equal to twice the area of the triangle AOC, and the product $BC \cdot OE$ is twice the area of the triangle BOC. Moreover, these triangles are equal to each other. Thus, the moments created by the springs (1) and (2) relative to the axis O, are always equal to each other.

The elasticity c_1 of the spring (1), reduced to the axis O or, in other words, the coefficient of recovery of the positional moment M_1 set up by the force of STAT spring, is

$$c_1 = \frac{dM_1}{d\alpha} = \frac{d(S \cdot AO \cdot OC \cdot \sin \alpha)}{d\alpha} = S \cdot AO \cdot OC \cdot \cos \alpha.$$

To derive an expression for the reduced elasticity c_2 of another spring, it is necessary to take the derivative of the absolute magnitude of the moment M_2 , created by the stress of this spring, in accordance with the argument $\pi - \alpha$, in view of the fact that one of the terms of the definition of the coefficient of recovery is that the rotation about the axis of the moment is positive in the direction opposite to the action of the moment. Therefore,

$$c_2 = \frac{d|M_2|}{d(\pi - \alpha)} = \frac{d(S \cdot OB \cdot OC \cdot \sin \alpha)}{d(\pi - \alpha)} = \frac{d[S \cdot OB \cdot OC \cdot \sin(\pi - \alpha)]}{d(\pi - \alpha)} = \\ = S \cdot OB \cdot OC \cdot \cos(\pi - \alpha) = -S \cdot OB \cdot OC \cdot \cos \alpha.$$

From this it is also clear that, of the springs (1) and (2) the only spring with positive reduced elasticity is the one for which the angle of contraction is less than $\frac{\pi}{2}$ since, if this angle is greater than $\frac{\pi}{2}$, its elasticity, reduced to the 0 axis, is negative.

This suggests an additional design, whose advantage lies in the fact that it permits an adjustment of the total coefficient of recovery to any desired magnitude, at will (but, of course, within some specific range), while the total coefficient of recovery may arbitrarily be made positive or negative. For this purpose, all that is necessary is that the levers OA and OB are able to rotate opposite each other, and that they are mounted at the required angle β (Fig. 61) relative to each other. Then, the moment of one spring is $S \cdot OA \cdot OC \sin \alpha_2$. The total moment M is

$$M = S \cdot OA \cdot OC (\sin \alpha_1 - \sin \alpha_2) = S \cdot OA \cdot OC \cdot 2 \cos \frac{\alpha_1 + \alpha_2}{2} \sin \frac{\alpha_1 - \alpha_2}{2} = \\ = S \cdot OA \cdot OC \cdot 2 \cos \frac{\beta}{2} \sin \varphi,$$

where φ is the angle of rotation of the lever relative to its center position. Here, it is true, we do not obtain a constant coefficient of recovery, since the moment ^{STAT} is proportional to the sine of the angle φ and not to the angle φ itself, i.e., the coefficient of recovery is of the same type as in the common pendulum (and is nearly

constant when the φ angle is small). However, we are able to have a coefficient of any desired magnitude, relative to the size of the angle β . In that connection, depending on whether the angle β is smaller or larger than π , we obtain a positive or negative coefficient of recovery, respectively. In addition, we obtain either a moment opposite to the direction of deflection φ of the lever OC from its center position

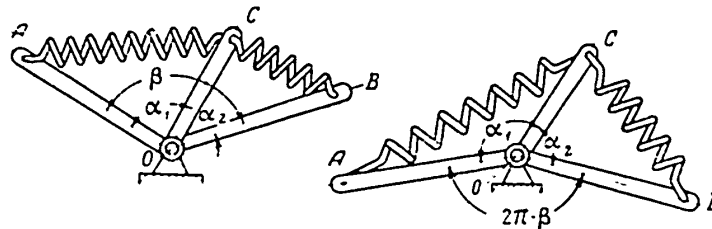


Fig.61 - Schematic Mechanism in Which the Total Elasticity of Two Springs Reduced to the Lever OC May be Rendered Positive or Negative at Will

tion, or a moment in the same direction.

It would be possible to derive a great number of additional examples, similar to that indicated above, but the above examples are adequate to illustrate the general propositions set forth in this Chapter.

Of course, in the general case of various combinations of other types of functions $F(x)$ and $f(y)$, it is not always possible to reproduce eq.(7.15) by means of only the usual simple levers. In a number of cases, it is necessary to use shaped cams, the use of which will apparently make it possible to satisfy eq.(7.15) in nearly all cases.

Let us note further that cases in which the elasticity of a spring changes in sign, upon reduction, are encountered with extreme rarity in the mechanisms of springs. In addition, if the mechanism of an instrument contains forces with negative reduced coefficients of recovery, allowance is always made for other positional forces having positive reduced coefficients of recovery (larger in absolute magnitude than the negative forces), so that the overall total coefficient of recovery is

always positive; otherwise, the position of equilibrium of the mechanism at different instrument readings would not be stable.

As a rare example of the installation of a spring with a negative reduced coefficient of recovery in the given instrument, we could conceive instruments with what is known as "force" compensation of temperature (see Figs. 87, 88, and 89 in Chapter XI). But even here, the object of such mounting of a spring is not an effort to reduce the overall rigidity of the system so as to give the pickup element greater play, but an effort to compensate the temperature errors of an instrument, since the spring is of the bimetal type.

It should further be noted that a result analogous to that obtained in Fig. 49 and 53 by reaction of a centrifugal force and an elastic force, may also be obtained by reaction of any positional force of the negative reduced coefficient of recovery, reduced to any part of the mechanism, and an ordinary elastic force acting on that part (i.e., a force having a positive coefficient of recovery).

Thus, if we use a spring contact (as in Fig. 53) to produce this latter force, and, by means of a specific transmission mechanism, connect the part in contact therewith to some other force, also elastic, but select a transmission mechanism such that the reduced coefficient of recovery of this latter force is negative and approximately equal, in absolute magnitude, to the stiffness of the elastic contact, the device thus developed will have the following characteristic: A very small additional external force, acting on the part in question (for example, a very small inclination of the device, if the part is a horizontal pendulum with a vertical axis), may cause considerable pressure at the contact. It would appear that devices of this type might find application in sensitive electric relays.

Section 2. Parallel and Series Connection of Spring Elements

If a mechanism contains several spring elements working together, their to ^{STAT} elasticity will depend on the nature of the connection between these elements and

with other elements of the instrument. This connection may be either of the parallel or series type.

Spring elements are considered connected in parallel if one end of each such component is rigidly fixed to the foundation relative to which measurement is made, while the other ends act in combination on the given mechanism, tending to displace it either in the same or in opposite directions. In certain systems of this type, all the spring components act directly on a single point of the mechanism, while in others spring components may act at different points in the mechanism. In the latter case, in order to determine total elasticity, it is necessary to reduce the elasticity of all the individual components to a single specific point in the mechanism.

As indicated before, the total stiffness of such a system reduced to some specific point (or axis) of the mechanism, is equal to the algebraic sum of all the stiffnesses of the individual spring components, reduced to that point.

The thermal error ΔP_t of such a system in any position, expressed in units of the force measured, is equal to

$$\Delta P_t = (P_{1 \text{ red}} \beta_1 + P_{2 \text{ red}} \beta_2 + P_{3 \text{ red}} \beta_3 + \dots) \Delta t, \quad (7.21)$$

where $P_{1 \text{ red}}$, $P_{2 \text{ red}}$, and $P_{3 \text{ red}}$ are, respectively, the forces P_1 , P_2 and P_3 of specific spring components corresponding to the given position of the mechanism, and

The factors β_1 , β_2 , and β_3 are, respectively, the temperature coefficients of the moduli of elasticity of the materials of which the elastic components are composed.

The signs of the various terms in the reduced formula are those obtainable at a given reduced force. In other words, we are dealing with an algebraic sum.

In units of travel x of the point to which the forces are reduced, the thermal error under examination is apparently

$$\Delta x_t = \Delta P_t \frac{dx}{d(P_{1 \text{ red}} + P_{2 \text{ red}} + P_{3 \text{ red}} + \dots)}. \quad (7.22)$$

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A special case of parallel connection of spring components is the combined working of two elements connected by a tube, where the purpose of these components is pressure measurement (i.e., pressure boxes, Bourdon tubes, bellows, etc.), in which the entire system is filled with some liquid serving to transmit pressure. In such a system, one of the spring components is merely a receiver and divider, accepting the pressure to be measured, which is then transferred through the tube to the second element, now serving as the pickup elastic element of the instrument in question.

An example of such a system is the obsolete type of aircraft oil pressure gage, in which, to eliminate fouling or thickening of the oil, a special pick-up element was provided. Such an instrument consisted of the following components: a) pickup; b) measuring element (the pressure gage proper); and c) the tube linking pickup and measuring element.

The most important part of the pickup is the sylphon, whose inside space is connected by a tube to the inside of the pressure gage Bourdon tube. The entire interior space of this system, consisting of sylphon, Bourdon tube, and tube line, is filled with the liquid used to transmit pressure, usually toluene. The sylphon (bellows) is housed in the receiver casing, which is mounted to that point of the oil line of the aircraft where the pressure has to be measured.

In such a system, the oil pressure counteracts the total force of elasticity of the Bourdon tube and bellows. What actually happens is that when the Bourdon tube straightens out under the effect of excess internal pressure, its volume increases. This is natural since the work performed by this pressure consists exactly in this. The total volume of the pressure-transmitting fluid may be regarded as constant for all practical purposes (we do not here discuss thermal expansion of the liquid, which will be treated below). Therefore, as the volume of the Bourdon tube increases, the volume of the receiver bellows must decline in equal measure, i.e., STAT bellows must undergo compression, and its elasticity must be overcome in order for this to happen.

Thus, if the measured differential pressure, i.e., the difference between the pressure around the bellows and the pressure around the Bourdon tube, is equal to p , then only some portion p_1 of this pressure (the larger part, of course), is consumed for straightening out the Bourdon tube, while the rest of the pressure $p - p_1$ is consumed in compressing the bellows. Therefore, the bellows is affected by a differential pressure equal to

$$p_2 = -(p - p_1).$$

The minus sign is used here because we assume p_2 to be positive when the pressure within a bellows is greater than that around it.

Thus,

$$p = p_1 - p_2.$$

When the differential pressure p being measured varies by dp , we have

$$dp = dp_1 - dp_2. \quad (7.23)$$

On the other hand, we know that

$$dV_c + dV_b = 0,$$

where V_b is the inside volume of the Bourdon tube,

V_c is the inside volume of the bellows.

The last equation above may be written as

$$dp_1 \left(\frac{dV_b}{dp_1} \right) + dp_2 \left(\frac{dV_c}{dp_2} \right) = 0. \quad (7.24)$$

Here $\frac{dp_1}{dV_b}$ is the ratio of the increase in excess pressure within the Bourdon tube to the corresponding increase in its volume. This magnitude is virtually ^{STAT}stant over the entire range of pressures measured by the Bourdon tube, i.e., it is

equal to $\frac{\Delta P_1}{\Delta V_b}$.

V. I. Feodos'yev, in his paper on the theory of the Bourdon tube, provides a formula for calculating that magnitude. In industry, this is usually determined by experiment.

The quantity $\frac{dp_2}{dV_c}$ is the ratio of the increase in differential pressure acting on the bellows, to the corresponding increase in the bellows volume. This magnitude equals

$$\frac{dp_2}{dV_c} = \frac{dp_2}{dy} \frac{dy}{dV_c} = \frac{dp_2}{dy} : \frac{dV_c}{dy}, \quad (7.25)$$

where y is the deflection of the free moving base of the bellows under the effect of the differential pressure p_2 .

Thus, in this last equation, the expression $\frac{dp_2}{dy}$ determines the elasticity of the syphon, and $\frac{dV_c}{dy}$ is the ratio of variation in the volume V_c of the bellows to the change in its deflection y . This value is, for the most part, only slightly different from

$$\frac{dV_c}{dy} \approx \frac{1}{3} \pi (R^2 + Rr + r^2),$$

where R is the radius of the outer periphery of the bellows,

r is the radius of its narrowest portion.

A combined solution of eqs. (7.23) and (7.24) gives us

$$\Delta p_1 = \Delta p \frac{\frac{dV_c}{dp_2}}{\frac{dV_c}{dp_2} + \frac{dV_b}{dp_1}}; \quad (7.26)$$

$$\Delta p_2 = -\Delta p \frac{\frac{dV_b}{dp_1}}{\frac{dV_c}{dp_2} + \frac{dV_b}{dp_1}} \quad (7.27) \text{ STAT}$$

It can be demonstrated that it is better to render magnitude $\frac{\frac{dV_c}{dp_2}}{\frac{dV_c}{dp_2} + \frac{dV_b}{dp_1}}$ as close to unity as possible, since this will result in an instrument with a lower thermal error. This requires that the quantity $\frac{dV_c}{dp_2}$ be as large as possible relative to $\frac{dV_b}{dp_1}$.

It should be noted, however, that $\frac{dV_b}{dp_1}$ is usually very low in Bourdon tubes. However, the magnitude of $\frac{dV_c}{dp_2}$ may be rendered quite large [see eq.(7.25)]. Actually, the elasticity $\frac{dp_2}{dV}$ of the syphon may be rendered quite small, since it has an adequate number of corrugations, and the thickness of the bellows wall may be very slight in view of the fact that the differential pressure $p - p_1$ may be quite low in the case that $\frac{dV_c}{dp_2}$ is sufficiently greater than $\frac{dV_b}{dp_1}$, as may be seen from the terms in eq.(7.27).

In calculating the temperature error of such a system, it is necessary to add to the error described by eq.(7.22), an error resulting from a variation in the volume of the pressure-transmitting fluid with varying temperature. Thus, if the total volume of the fluid in a bellows, in the Bourdon tube, and in the connecting tube has some average calculated temperature $V_{0,zh}$, then, if the temperature of the liquid varies by Δt_{zh} , the volume of the liquid, $V_{l,zh}$ will be

$$V_{l,zh} = V_{0,zh}(1 + \alpha_{zh}\Delta t_{zh}),$$

where α_{zh} is the thermal coefficient of the volumetric expansion of the liquid.

Thus, the increment ΔV_{zh} in the volume of the liquid is

$$\Delta V_{zh} = V_{0,zh}\alpha_{zh}\Delta t_{zh}.$$

Analogously, the volumes of the Bourdon tube, the tube line and the bellows vary only due to variations in their temperatures by the following amounts, respectively

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$$\Delta V_b = V_{o.b} \alpha_b \Delta t_b;$$

$$\Delta V_T = V_{o.T} \alpha_T \Delta t_T;$$

$$\Delta V_c = V_{o.c} \alpha_c \Delta t_c,$$

where

$$V_{o.b} + V_{o.T} + V_{o.c} = V_{o.zh}$$

Here $V_{o.b}$, $V_{o.T}$ and $V_{o.c}$ are, respectively, the volumes of the Bourdon tube, the tube conduit and the bellows, at the calculated temperature;

α_b , α_T and α_c are, respectively, the thermal coefficients of volumetric expansion of the materials of which the Bourdon tube, the tube conduit, and the bellows are composed;

Δt_b , Δt_T and Δt_c are, respectively, the variations in temperature of these components.

Since the thermal coefficient of expansion of the liquid is usually many times as large as the coefficients of volumetric expansion of the materials of which the spring elements and the tube line are composed, any variations in temperature will result in an additional expansion ΔV of the fluid, equal to

$$\Delta V = \Delta V_{zh} - (\Delta V_b + \Delta V_T + \Delta V_c).$$

This excess variation in the volume of the transmitting liquid results in an additional deflection of the syphon and Bourdon tube or, in other words, changes the differential pressure acting on the Bourdon tube. In this connection, some effect is also felt by variations in the elasticity of the pickup bellows, if its temperature has changed, relative to the calculated temperature, by some magnitude Δt_c .

The total magnitude of the error resulting from all these causes may be calculated, approximately, by the usual method. Let us determine the magnitude Δp by which the pressure being measured p must be varied, in order for the deflection of

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the Bourdon tube to change, regardless of any variations in the temperature of the Bourdon tube itself (Δt_b), variations in the temperature of the pressure-transmitting liquid by Δt_{ch} , and variation in the temperature of the bellows by Δt_c . In this connection let us assume, approximately, that if the deflection of the Bourdon tube does not change, neither does its volume (except for changes in volume due to changes in temperature).

Thus it is required that all excess change ΔV in the volume of the liquid is absorbed by changes in the volume of the bellows, i.e., by some change in the deflection, while the Bourdon tube should not vary in volume as a result. Thus, the differential pressure p_2 acting on the bellows must change in such a fashion that the bellows, first, must change its deflection by a magnitude Δy , equal to

$$\Delta y = \frac{\Delta V}{\left(\frac{dV_c}{dy}\right)}$$

[see symbols for eq.(7.25)].

In the second place, it is essential that this deflection is not affected by a change in the elasticity of the siphon as its temperature changes by Δt_c .

We may consciously break down this phenomenon into two stages:

a) The bellows deflects so that its volume varies by ΔV , while the temperature of the bellows remains constant;

b) The temperature of the bellows and of the differential pressure acting thereon must change, correspondingly, in such a fashion that the deflection of the bellows undergoes no further change.

The first stage requires a variation $\Delta p_2'$, in the differential pressure p_2 , equal to

$$\Delta p_2' = \Delta y \left(\frac{dp_c}{dy}\right)_{cp} = \frac{\Delta V}{\left(\frac{dV_c}{dy}\right)} \left(\frac{dp_2}{dy}\right)_{cp} \approx \frac{\Delta V}{\left(\frac{dV_c}{dy}\right)} \left(\frac{dp_2}{dy}\right) = \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)}$$

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where $\left(\frac{dp_2}{dy}\right)_{cp}$ is the average elasticity of the bellows when deflected by Δy .

In view of the fact that this deflection is small, the quantity $\left(\frac{dp_2}{dy}\right)_{cp}$ may be substituted, in first approximation, by the elasticity $\frac{dp_2}{dy}$ of the bellows, at a pressure of p_2 .

For the second stage of differential pressure acting on the bellows, there has to be a variation by $\Delta p_2''$, equal to

$$\Delta p_2'' = -\beta_c(p_2 + \Delta p_2') \Delta t_c = -\beta_c p_2 \Delta t_c - \beta_c \Delta p_2' \Delta t_c,$$

and, in the sum total,

$$\Delta p_2 = \Delta p_2' + \Delta p_2'' = \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)} - \beta_c p_2 \Delta t_c - \beta_c \Delta p_2' \Delta t_c. \quad (7.26)$$

This phenomenon may be broken down into two other stages:

- a) The temperature of the bellows and the pressure acting thereon vary, corresponding, by such an amount that the deflection of the bellows undergoes no change;
- b) The pressure acting on the bellows changes further to cause it to acquire the necessary deflection Δy .

For the first stage, the pressure p_2 must change by the value $(\Delta p_2)_I$, equal to

$$(\Delta p_2)_I = -\beta_c p_2 \Delta t_c.$$

For the second stage, the pressure acting on the bellows must change by $(\Delta p_2)_{II}$ being equal to

$$(\Delta p_2)_{II} = \Delta y \left(\frac{dp_2}{dy}\right)_{t_c} = \Delta y \left(\frac{dp_2}{dy}\right) (1 - \beta_c \Delta t_c) = \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)} - \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)} \beta_c \Delta t_c.$$

Adding $(\Delta p_2)_I$ and $(\Delta p_2)_{II}$, we again emerge with eq.(7.28).

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Ignoring in eq.(7.28) the third term on the right side as a small second-order

magnitude, we obtain

$$\Delta p_2 = \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)} - \beta_c p_2 \Delta t_c.$$

In order for the Bourdon tube not to change its deflection under the indicated conditions, a change Δp_1 is required in the differential pressure p_1 acting thereon, and this comes to

$$\Delta p_1 = -\beta_b p_1 \Delta t_b,$$

where β_b is the thermal coefficient of the modulus of elasticity of the material of which the tube is composed.

Thus, to satisfy the conditions we have posed in accordance with eq.(7.23), it is necessary that the pressure to be measured change by the following sum:

$$\Delta p = \Delta p_1 - \Delta p_2 = -\beta_b p_1 \Delta t_b - \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)} + \beta_c p_2 \Delta t_c. \quad (7.29a)$$

If only the temperature changes without an accompanying change in the pressure p , the instrument readings will vary approximately by the reciprocal, i.e., by Δp_p , which comes to

$$\Delta p_p \approx \beta_b p_1 \Delta t_b + \frac{\Delta V}{\left(\frac{dV_c}{dp_2}\right)} - \beta_c p_2 \Delta t_c. \quad (7.29)$$

It should be noted, in this connection, that the third term on the right side of the derived expression often has the same sign as the first two terms. Actually, both eq.(7.24) and the nature of the work done by the device show that the pressure p_2 is very often negative. STAT

Equation (7.29) makes clear why the magnitude of $\frac{dV_c}{dp_2}$ must be as large as possible, since an increase in this value reduces the error Δp_p . In this connection,

it should be also remembered that this increase in the expression $\frac{\Delta V_c}{dp_2}$ serves further to reduce the magnitude of p_2 , as demonstrated in eq.(7.27), i.e., it reduces the magnitude not only of the second term in eq.(7.29), but of the third as well.

Spring components may also be linked in series. In this arrangement, only one of the spring elements is fixed, at one end, to the base from which the deflection of the entire system of spring elements is measured. One end of the second elastic component is attached to the end of this first one, a third being attached to the end of the second, etc.

As an example of such a system, we may cite a battery of Weedy boxes in which only one, an end box, is attached to the base of the instrument, and the motion of the rigid center of the other extreme box is measured.

Let us, in such a box, measure the displacement of the free end of the last of the elastic elements (i.e., the end not connected to the preceding part). This means that the total deflection of all the elastic elements changes.

In this case, the elasticity of the system is, of course, always less than the elasticity of each of the elastic parts individually. In fact, when the load on dP changes (in a case where the concentrated load is measured), the individual elastic elements are deflected, respectively, by $dx_1, dx_2, dx_3 \dots$ etc. Therefore the total elasticity of the system c_c when measuring concentrated loads, is

$$c_c = \frac{dP}{dx_1 + dx_2 + dx_3 + \dots} = \frac{dP}{\sum (dx_i)}$$

while the elasticity S_c , when measuring the distributed load, will be

$$S_c = \frac{d\bar{P}}{\sum (dx_i)},$$

in which case the deflection dx_1 of one element becomes:

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At concentrated load

$$dx_1 = \frac{dP}{c_1}$$

(where c_i is the elasticity of this element under concentrated load),

At distributed load

$$dx_i = \frac{dp}{S_i}$$

(where S_i is the elasticity of the element under distributed load).

On this basis, the total elasticity of the system is expressed by the equation:

At concentrated load

$$\left. \begin{aligned} c_c &= \frac{1}{\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \dots} = \frac{1}{\sum \left(\frac{1}{c_i} \right)}, \\ S_i &= \frac{1}{\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \dots} = \frac{1}{\sum \left(\frac{1}{S_i} \right)}. \end{aligned} \right\} \quad (7.30)$$

Thus, if a battery consists of two diaphragms whose elasticities are, respectively, S_1 and S_2 , the elasticity of the entire battery will be

$$S_c = \frac{S_1 S_2}{S_1 + S_2}.$$

The temperature error of such a block of elastic elements connected in series, expressed in units of total deflection of the block, becomes under concentrated load, P

$$\Delta x \approx P \Delta t \left(\frac{\beta_1}{c_1} + \frac{\beta_2}{c_2} + \frac{\beta_3}{c_3} + \dots \right)$$

(where $\beta_1, \beta_2, \beta_3 \dots$ are the thermal coefficients of the moduli of elasticity of the materials of these elements),

and under distributed load, p

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$$\Delta x \approx p \Delta t \left(\frac{\beta_1}{S_1} + \frac{\beta_2}{S_2} + \frac{\beta_3}{S_3} + \dots \right).$$

In units of the force being measured P , or of the pressure p , this thermal error is approximately

$$\left. \begin{aligned} \Delta P &\approx \Delta x c_c \approx P \Delta t \frac{\frac{\beta_1}{c_1} + \frac{\beta_2}{c_2} + \frac{\beta_3}{c_3} + \dots}{\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \dots} ; \\ \Delta p &\approx \Delta x S_c \approx p \Delta t \frac{\frac{\beta_1}{S_1} + \frac{\beta_2}{S_2} + \frac{\beta_3}{S_3} + \dots}{\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \dots} \end{aligned} \right\} \quad (7.31)$$

In conclusion let us determine the magnitude of the total mean effective area of several diaphragms connected in series.

The boxes for measuring the pressure consist of two diaphragms soldered together. In the majority of cases, the two diaphragms are of identical effective area, so that the effective area of the box is equal to the effective area of one diaphragm. Similarly, the blocks of several boxes connected in series are usually made up of individual cells, so that the effective area of such a block is equal to the effective area of a single diaphragm.

Cases may be encountered, however, in which the pressure cell consists of two unequal diaphragms (i.e., if the stiff centers of the diaphragms differ), or if the unit consists of diaphragms that are not identical. In this case, it is obvious that we must take some average quantity F_{cp} of the effective areas of the diaphragms. This value is not difficult to compute.

In accordance with the definition derived above

$$F_{cp} = \frac{dP}{dp},$$

where dP and dp are variations in the concentrated and distributed stresses applied to the unit, of such a nature that the total deflection of the unit does not change.

However, the force dP , if it were acting individually, would change the deflection of the unit by a value dx_p , equal to

$$\begin{aligned} dx_p &= (dx_1)_p + (dx_2)_p + (dx_3)_p + \dots = \\ &= dP \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \dots \right), \end{aligned}$$

where $(dx_1)_p$, $(dx_2)_p$, $(dx_3)_p$ are the deflections of various diaphragms in the unit, c_1 , c_2 , and c_3 are, correspondingly, the stiffnesses of these diaphragms when under concentrated load.

In view of the fact that the equation for the effective area of a diaphragm, in accordance with this definition, reads

$$c = FS,$$

where S is the stiffness of the diaphragm under distributed load (i.e., the magnitude $S = \frac{dp}{dx}$, which we usually know for the given diaphragm) and F is its effective area, then

$$dx_p = dP \left(\frac{1}{F_1 S_1} + \frac{1}{F_2 S_2} + \frac{1}{F_3 S_3} + \dots \right).$$

Identically the same distributed pressure dp , if it were acting individually, would change the total deflection of the unit by dx_p , which comes to

$$\begin{aligned} dx_p &= (dx_1)_p + (dx_2)_p + (dx_3)_p + \dots = \\ &= dp \left(\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \dots \right). \end{aligned}$$

However, in the given case we have $dx_p = dx_p$, so that we obtain, from the last two equations

$$F_{cp} = \frac{dP}{dp} = \frac{\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \dots}{\frac{1}{F_1 S_1} + \frac{1}{F_2 S_2} + \frac{1}{F_3 S_3} + \dots} \quad \text{STAT} \quad (7.32)$$

In the special case of a single unit, consisting of two diaphragms whose effective areas are F_1 and F_2 , we obtain a total average effective area per box of

$$F_{cp} = \frac{\frac{1}{S_1} + \frac{1}{S_2}}{\frac{1}{S_1 F_1} + \frac{1}{S_2 F_2}} = \frac{F_1 F_2 (S_1 + S_2)}{F_1 S_1 + F_2 S_2} \quad (7.33)$$

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CHAPTER VIII

BIMETAL STRIPS

Bimetal strips, installed in some link of the multiplying mechanism, are in wide use for compensating temperature errors in many mechanisms. In addition, bimetal parts are used as pickup elements in certain thermometers and thermostats. In this Chapter we will present the derivation of the major equations for bimetals.

A bimetal strip is, of course, a strip consisting of two strips of different metals differing in their coefficients of thermal expansion, attached to each other so rigidly (sometimes even by welding) that they cannot separate or undergo relative displacements despite the difference in expansion (or contraction) due to variations in temperature.

We also know that such a bimetal strip deflects on changes in the temperature, since one strip expands or contracts more than the other, while they are unable to become detached. It is clear that these two conditions are compatible only if the bimetal strip is free to bend toward that of its components that expands less (if both are undergoing elongation), or toward that which contracts more (if both are undergoing contraction).

Section 1. Basic Formula for Deflection of Straight Bimetal Strip

Figure 62 presents a longitudinal section through a bimetal strip. Let the modulus of elasticity of one metal be E_1 , the thermal coefficient of its linear elongation α_1 , and its thickness h_1 . The modulus of elasticity of the other metal

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is E_2 , its thermal coefficient of linear expansion α_2 , and its thickness h_2 . The thickness of the two metal strips is identical, and is denoted by b .

It is obvious that when the bimetal strip is bent by changes in the temperature,

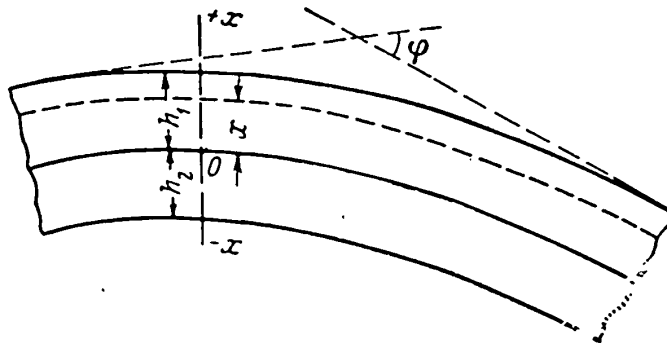


Fig.62 - Thermal Deflection of Bimetal Strip

the elongation in some layers of the strip will be greater than those that would result only from a variation in temperature, while in others it will be less. Correspondingly, tensile stresses will arise in certain layers and compressive stresses in others.

If the junction between the two components, whose length is l , becomes elongated during a change in temperature by a magnitude Δl_1 , and the strip then deflects by an angle φ , the overall elongation of a layer x distances away from the surface of the junction will be $\Delta l + x\varphi$. If x is assumed as positive in the direction of the metal with the higher coefficient of expansion and negative in the direction of the metal with the lower coefficient, this expression is valid for any layer in both metals.

Note: It is assumed, in this derivation, that the flat transverse cross section of the bimetal strip, normal to the surface of the junction, remain flat and normal to the surface of the junction even after the strip has undergone bending, and that the strip bends uniformly in the arc of a circle.

Under these conditions, the difference between the resultant elongation of any layer of the first metal (with the larger coefficient of expansion α_1) and its elongation due exclusively to a variation in temperature, is $\Delta l + x\varphi - l\alpha_1\Delta t$, where Δt is the variation in temperature, while in the second metal this difference is $\Delta l + x\varphi - l\alpha_2\Delta t$.

This fundamental layer (of a thickness dx) of the first metal requires a tensile stress of

$$\frac{\Delta l + x\varphi - l\alpha_1\Delta t}{l} E_1 b dx,$$

while, for the second metal, the stress is

$$\frac{\Delta l + x\varphi - l\alpha_2\Delta t}{l} E_2 b dx.$$

All such stresses acting in cross sections of the strip constitute external stresses relative to the segments of the strip adjacent to the given section, and must therefore be in equilibrium as a whole. The first condition for equilibrium is that the sum of all the stresses must equal zero

$$\int_0^{h_1} \frac{\Delta l + x\varphi - l\alpha_1\Delta t}{l} E_1 b dx + \int_{-h_2}^0 \frac{\Delta l + x\varphi - l\alpha_2\Delta t}{l} E_2 b dx = 0. \quad (8.1)$$

The sum of the moments of all stresses relative to some point (which we take to be the point of intersection of the surface of the couple and the plane of the cross section) must be equal to zero, i.e.,

$$\int_0^{h_1} \frac{\Delta l + x\varphi - l\alpha_1\Delta t}{l} E_1 b x dx + \int_{-h_2}^0 \frac{\Delta l + x\varphi - l\alpha_2\Delta t}{l} E_2 b x dx = 0. \quad (8.2)$$

From eqs.(8.1) and (8.2) we are able to determine Δl and φ . By integrating ^{STAT} both equations, and abridging both equations by $\frac{b}{l}$, we obtain

$$E_1 \left(\Delta l h_1 + \frac{h_1^2}{2} \varphi - l h_1 \alpha_1 \Delta t \right) + E_2 \left(\Delta l h_2 - \frac{h_2^2}{2} \varphi - l h_2 \alpha_2 \Delta t \right) = 0;$$

$$E_1 \left(\Delta l \frac{h_1^2}{2} + \frac{h_1^3}{3} \varphi - l \frac{h_1^2}{2} \alpha_1 \Delta t \right) + E_2 \left(-\Delta l \frac{h_2^2}{2} + \frac{h_2^3}{3} \varphi + l \frac{h_2^2}{2} \alpha_2 \Delta t \right) = 0.$$

From this we derive

$$\left. \begin{aligned} \Delta l (E_1 h_1 + E_2 h_2) + \varphi \frac{E_1 h_1^2 - E_2 h_2^2}{2} - l (E_1 h_1 \alpha_1 + E_2 h_2 \alpha_2) \Delta t &= 0; \\ \Delta l \frac{E_1 h_1^2 - E_2 h_2^2}{2} + \varphi \frac{E_1 h_1^3 + E_2 h_2^3}{3} - l \frac{E_1 h_1^2 \alpha_1 - E_2 h_2^2 \alpha_2}{2} \Delta t &= 0. \end{aligned} \right\} \quad (8.3)$$

Eliminating the unknown Δl from both these equations, we have

$$\begin{aligned} \varphi \left[\frac{(E_1 h_1^3 + E_2 h_2^3)(E_1 h_1 + E_2 h_2)}{3} - \frac{(E_1 h_1^2 - E_2 h_2^2)^2}{4} \right] = \\ = \frac{l}{2} [(E_1 h_1^2 \alpha_1 - E_2 h_2^2 \alpha_2)(E_1 h_1 + E_2 h_2) - \\ - (E_1 h_1 \alpha_1 + E_2 h_2 \alpha_2)(E_1 h_1^2 - E_2 h_2^2)] \Delta t. \end{aligned}$$

After the appropriate manipulations of this equation, we arrive at

$$\begin{aligned} \varphi [4 E_1 E_2 (h_1^3 h_2 + 2 h_1^2 h_2^2 + h_1 h_2^3) + 4 (E_1 h_1^2 - E_2 h_2^2)^2 - \\ - 3 (E_1 h_1^2 - E_2 h_2^2)^2] = 6 l E_1 E_2 h_1 h_2 (h_1 + h_2) (\alpha_1 - \alpha_2) \Delta t, \end{aligned}$$

from which we derive

$$\varphi = l \Delta t \frac{6 E_1 E_2 h_1 h_2 (h_1 + h_2) (\alpha_1 - \alpha_2)}{4 E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} = k_0 l \Delta t, \quad (8.4)$$

where

$$k_0 = \frac{6 E_1 E_2 h_1 h_2 (h_1 + h_2) (\alpha_1 - \alpha_2)}{4 E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2}.$$

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The magnitude of k_0 for each given bimetal strip is a constant. When the value

of $(h_1 + h_2)$ or, in other words, the total thickness of the strip, is constant, k_0 is of maximum value if the ratio between h_1 and h_2 is such that the expression in parentheses $(E_1 h_1^2 - E_2 h_2^2)$ becomes zero.

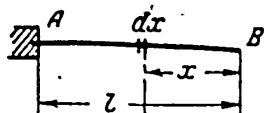


Fig.63 - Thermal Deflection of a Straight Bimetal Strip

This is readily confirmed if we write the expression for $\frac{1}{k_1}$, the reciprocal of k_0

$$\frac{1}{k_0} = \frac{2}{3} \frac{(h_1 + h_2)}{(z_1 - z_2)} + \frac{(E_1 h_1^2 - E_2 h_2^2)^2}{6 E_1 E_2 h_1 h_2 (h_1 + h_2) (z_1 - z_2)}$$

It is obvious that, when $h_1 + h_2$ is constant, it has its lowest value since the second term is reduced to zero.

In view of this fact, the thickness of each of the two components of the strip is usually so designed, in bimetal strip manufacture, that

$$\frac{h_1}{h_2} = \sqrt{\frac{E_2}{E_1}} \quad (8.5)$$

In this case eq.(8.4) undergoes a considerable simplification

$$\varphi = \frac{3}{2} \frac{(z_1 - z_2)}{(h_1 + h_2)} l \Delta t, \quad (8.6)$$

i.e.,

$$k_0 = \frac{3}{2} \frac{(z_1 - z_2)}{(h_1 + h_2)}$$

If we are dealing with a bimetal strip fixed at one end A (Fig.63), the path of the other end B, due to changes in the temperature (which we have to know to calculate the thermal compensation) may be calculated in the following manner:

Let us take some section of a bimetal strip of infinitesimal length dx , located at a distance x from the B end. A change of Δt in the temperature causes it to rotate through an angle of $k_0 \Delta t dx$. As a result, point B is displaced in a direction nor-

mal to AB by $df = \alpha k_0 \Delta t dx$. From this we find that the total deflection f of the B end of the strip is

$$f = \int_0^l k_0 \Delta t x dx = k_0 \Delta t \frac{l^2}{2}, \quad (8.7)$$

where l is the length of the strip AB

Section 2. Determining the Elongation of the Surface of a Couple

Eliminating the unknown φ from eq.(8.3), we may further determine the elongation, Δl of the surface of contact of the two metals at a variation Δt in the temperature (i.e., we determine the coefficient of thermal expansion of the strip).

This magnitude may be necessary, for example, in calculating the compensated bimetal balance of a watch mechanism. This calculation not only is to provide for the change in the deflection of the bimetal horns of the balance with any change in temperature, but also for the elongation of these horns. Excluding the unknown φ , we obtain

$$\begin{aligned} \Delta l \left[\frac{(E_1 h_1 + E_2 h_2)(E_1 h_1^3 + E_2 h_2^3)}{3} - \frac{(E_1 h_1^2 - E_2 h_2^2)^2}{4} \right] = \\ = l \left[\frac{(E_1 h_1 \alpha_1 + E_2 h_2 \alpha_2)(E_1 h_1^3 + E_2 h_2^3)}{3} - \frac{(E_1 h_1^2 \alpha_1 - E_2 h_2^2 \alpha_2)(E_1 h_1^2 - E_2 h_2^2)}{4} \right] \Delta t. \end{aligned}$$

The bracket on the left side of the equation is manipulated as in the preceding Section of our discussion. Thus, we have

$$\begin{aligned} \Delta l [4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2] = \\ = l \Delta t [4(E_1^2 h_1^4 \alpha_1 + E_1 E_2 h_1^3 h_2 \alpha_2 + E_1 E_2 h_1 h_2^3 \alpha_1 + E_2^2 h_2^4 \alpha_2) - \\ - 3(E_1^2 h_1^4 \alpha_1 - E_1 E_2 h_1^3 h_2 \alpha_2 - E_1 E_2 h_1 h_2^3 \alpha_1 + E_2^2 h_2^4 \alpha_2)] = \\ = l \Delta t [E_1^2 h_1^4 \alpha_1 + E_2^2 h_2^4 \alpha_2 + 4E_1 E_2 h_1^3 h_2 \alpha_2 + 4E_1 E_2 h_1 h_2^3 \alpha_1 + \\ + 3E_1 E_2 h_1^2 h_2^2 \alpha_1 + 3E_1 E_2 h_1^2 h_2^2 \alpha_2] = l \Delta t \left[E_1^2 h_1^4 \frac{\alpha_1 + \alpha_2}{2} + \right. \end{aligned}$$

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$$\begin{aligned}
& + E_1^2 h_1^4 \frac{a_1 - a_2}{2} + E_2^2 h_2^4 \frac{a_1 + a_2}{2} - E_2^2 h_2^4 \frac{a_1 - a_2}{2} + 4E_1 E_2 h_1^3 h_2 \frac{a_1 + a_2}{2} - \\
& - 4E_1 E_2 h_1^3 h_2 \frac{a_1 - a_2}{2} + 4E_1 E_2 h_1 h_2^3 \frac{a_1 + a_2}{2} + 4E_1 E_2 h_1 h_2^3 \frac{a_1 - a_2}{2} + \\
& + 8E_1 E_2 h_1^2 h_2^2 \frac{a_1 + a_2}{2} - 2E_1 E_2 h_1^2 h_2^2 \frac{a_1 + a_2}{2} = \\
& = l \Delta t \left\{ \frac{a_1 + a_2}{2} [E_1^2 h_1^4 + E_2^2 h_2^4 - 2E_1 E_2 h_1^2 h_2^2 + 4E_1 E_2 (h_1^3 h_2 + 2h_1^2 h_2^2 + h_1 h_2^3)] + \right. \\
& \quad \left. + \frac{a_1 - a_2}{2} [E_1^2 h_1^4 - E_2^2 h_2^4 - 4E_1 E_2 h_1^3 h_2 + 4E_1 E_2 h_1 h_2^3] \right\} = \\
& = l \Delta t \left\{ \frac{a_1 + a_2}{2} [4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2] + \right. \\
& \quad \left. + \frac{a_1 - a_2}{2} [E_1^2 h_1^4 - E_2^2 h_2^4 - 4E_1 E_2 h_1 h_2 (h_1^2 - h_2^2)] \right\}.
\end{aligned}$$

from which we obtain

$$\frac{\Delta l}{l} = \left\{ \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2} \frac{E_1^2 h_1^4 - E_2^2 h_2^4 - 4E_1 E_2 h_1 h_2 (h_1^2 - h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \right\} \Delta t. \quad (8.8)$$

Obviously, the expression in braces is the thermal coefficient of elongation of the surface of contact between the two metals (hereinafter referred to as α_b).

In case $E_1 h_1^2 = E_2 h_2^2$ [the ratio of the thicknesses, h_1 and h_2 , of metal strips is usually recommended in accordance with eq.(8.5)], we have

$$\alpha_b = \frac{a_1 + a_2}{2} - \frac{a_1 - a_2}{2} \frac{h_1 - h_2}{h_1 + h_2}. \quad (8.9)$$

Section 3. Derivation of Equations for Curved Bimetal Strips

Equations (8.4), (8.6), (8.8) and (8.9) have been derived for flat bimetal strips. If the strip is curved, slightly different formulas should result. The fact is that in our derivation of the equations in (8.3), we abridged eqs.(8.1) and (8.2) by the value $\frac{b}{l}$. However, along some segment of the strip, dl in length (measuring along the contact surface) where the strip has a radius of curvature ρ ,

different layers of the strip vary in length (i.e., they are not identical in length dl). But a derivation of accurate equations is very complex.

However, the errors due to the fact that, instead of a more precise equation

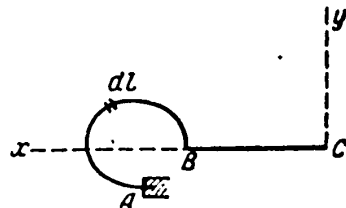


Fig.64 - Re Calculations for Thermal Deflection of Curved Bimetal Strip

for the curved strip, we use eqs.(8.4), (8.6), (8.8), and (8.9) are not large, the more so as since usually the thickness of the strip is negligible and therefore the ratio of the radius of its curvature to its thickness is quite large (meaning that the ratio of the lengths of the two metals is almost unity). Therefore we may for a

curved strip as well, make use (within the limits of approximation) of equations derived for a straight strip. In this case, however, it is better to rewrite eqs.(8.4) and (8.6) in the following form:

$$d\varphi_t - d\varphi_0 = \Delta(d\varphi) = k_0 dl \Delta t \quad (8.10)$$

or

$$\frac{1}{\rho_t} - \frac{1}{\rho_0} = k_0 \Delta t, \quad (8.10a)$$

where $\frac{1}{\rho_0} = \frac{d\varphi_0}{dl}$ is the initial radius of curvature of the bimetal strip (before heating), measured along the contact surface, while $\frac{1}{\rho_t} = \frac{d\varphi_t}{dl}$ is the radius of its curvature after heating.

In eqs.(8.10) and (8.10a), the terms $d\varphi_0$, $d\varphi_t$, ρ_0 , and ρ_t are deemed positive if the center of curvature of the given segment lies on the side of the contact surface occupied by the metal with the lower coefficient of elongation (α_2). However, if the center of curvature is on the other side, these terms must be deemed negative in eqs.(8.10) and (8.10a).

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Using eq.(8.10), it is easy to determine the linear deflection of the end of a

bimetal strip (the other end being fixed) curved in a specific shape, or the end of a lever attached to the end of such a strip.

For example, let us suppose that the strip is bent in a curve AB (Fig.64) and the lever BC is fastened to the end B. Then, taking C as our origin of coordinates, and taking the axes of the x and y coordinates in any two mutually perpendicular directions, one may determine the displacement of the point C in these two directions when the bimetal strip AB is deformed, due to a variation in temperature.

In point of fact, a flexure of a segment of the strip of a length dl , by an angle $\Delta(d\varphi)$ induces the following displacements of df_x and df_y of the point C in the directions Ox and Oy :

$$\begin{aligned} df_x &= y \Delta(d\varphi) = k_0 \Delta t y dl; \\ df_y &= -x \Delta(d\varphi) = -k_0 \Delta t x dl, \end{aligned}$$

where x and y are the coordinates of the segment under examination.

From this we obtain

$$\left. \begin{aligned} f_x &= k_0 \Delta t \int_A^B y dl; \\ f_y &= k_0 \Delta t \int_A^B x dl. \end{aligned} \right\} \quad (8.11)$$

In the equations of (8.11), integrations must be performed for the entire curved line from A to B.

Here the following is to be noted. If we multiply and divide the right sides of eqs.(8.11) by L, where L is the length of the developed strip from A to B, we obtain

$$\left. \begin{aligned} f_x &= k_0 L \Delta t \frac{\int_A^B y dl}{L} = k_0 L \Delta t Y; \\ f_y &= -k_0 L \Delta t \frac{\int_A^B x dl}{L} = -k_0 L \Delta t X, \end{aligned} \right\} \quad (8.12) \quad \text{STAT}$$

where X and Y are the coordinates of the center of gravity of the developed shape of the strip.

Thus, when the strip is bent by thermal effects, the point C will move approximately in the direction of the arc of a circle described around the given center of gravity (here we say "approximately" since this center of gravity itself always undergoes a slight displacement under gradual deflection of the strip).

With this rule as a basis, it is easy to determine by eye the approximate direction of the movements of the terminal point C of the pickup element when a bimetal strip is subjected to deflection due to temperature.

Section 4. Relative Elongation of a Bimetal Strip on Deflection by Heat

Equations (8.4) and (8.8) permit determining the relative elongations in various layers of a bimetal strip causing tension in these layers (for calculation of the permissible stress, if this is required). It is obvious that the total relative elongation of a layer x distances from the surface of contact, will be

$$\frac{\Delta l}{l} = \left\{ \frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_1 - \alpha_2}{2} \frac{E_1^2 h_1^4 - E_2^2 h_2^4 - 4E_1 E_2 h_1 h_2 (h_1^2 - h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \right\} \Delta t + x \frac{\varphi}{l}.$$

If we subtract, from the total elongation, the thermal elongation which in one metal is

$$\left(\frac{\Delta l}{l} \right)_1 = \alpha_1 \Delta t,$$

and in the other is

$$\left(\frac{\Delta l}{l} \right)_2 = \alpha_2 \Delta t,$$

we obtain the following elongation, producing a stress in the material:

In the first metal

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$$\frac{\Delta l - \Delta l_1}{l} = -\frac{\alpha_1 - \alpha_2}{2} \left\{ 1 - \frac{E_1^2 h_1^4 - E_2^2 h_2^4 - 4E_1 E_2 h_1 h_2 (h_1^2 - h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \right\} \Delta t + x \frac{\varphi}{l},$$

And in the second metal

$$\frac{\Delta l - \Delta l_t}{l} = \frac{\alpha_1 - \alpha_2}{2} \left\{ 1 + \frac{E_1^2 h_1^4 - E_2^2 h_2^4 - 4E_1 E_2 h_1 h_2 (h_1^2 - h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \right\} \Delta t + x \frac{\gamma}{l}.$$

After conversion we obtain, for the first metal,

$$\frac{\Delta l - \Delta l_t}{l} = \left[-(\alpha_1 - \alpha_2) \frac{E_2^2 h_2^4 + 4E_1 E_2 h_1^3 h_2 + 3E_1 E_2 h_1^2 h_2^2}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} + x k_0 \right] \Delta t$$

and, for the second,

$$\frac{\Delta l - \Delta l_t}{l} = \left[(\alpha_1 - \alpha_2) \frac{E_1^2 h_1^4 + 4E_1 E_2 h_1^3 h_2 + 3E_1 E_2 h_1^2 h_2^2}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} + x k_0 \right] \Delta t.$$

At the extreme surfaces of the two metals, where the tension may be greatest, we have

For the first metal

at the contact surface

$$\frac{\Delta l - \Delta l_t}{l} = -(\alpha_1 - \alpha_2) \frac{E_2^2 h_2^4 + 4E_1 E_2 h_1^3 h_2 + 3E_1 E_2 h_1^2 h_2^2}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \Delta t; \quad (8.13)$$

and at the outside surface, where $x = h_1$ (the conversions are not shown here, due to their simplicity).

$$\frac{\Delta l - \Delta l_t}{l} = -(\alpha_1 - \alpha_2) \frac{E_2^2 h_2^4 - 2E_1 E_2 h_1^3 h_2 - 3E_1 E_2 h_1^2 h_2^2}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \Delta t. \quad (8.14)$$

For the second metal

at the contact surface

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$$\frac{\Delta l - \Delta l_t}{l} = (\alpha_1 - \alpha_2) \frac{E_1^2 h_1^4 + 4E_1 E_2 h_1^3 h_2 + 3E_1 E_2 h_1^2 h_2^2}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2} \Delta t; \quad (8.15)$$

and at the outside surface where $x = -h_2$

$$\frac{\Delta l - \Delta l_t}{l} = (\alpha_1 - \alpha_2) \frac{E_1^2 h_1^4 - 2E_1 E_2 h_1 h_2^3 - 3E_1 E_2 h_1^2 h_2^2}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)} \Delta t. \quad (8.16)$$

From this it is clear that, in both metals, the absolute magnitude of the elongations under examination, and consequently of the stresses, are greatest at the point of junction. In the case of $E_1 h_1^2 = E_2 h_2^2$, we have,

For the first metal
at the contact surface

$$\begin{aligned} \frac{\Delta l - \Delta l_t}{l} &= -(\alpha_1 - \alpha_2) \frac{4E_1 E_2 (h_1^3 h_2 + h_1^2 h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2} \Delta t = \\ &= -(\alpha_1 - \alpha_2) \frac{h_1}{h_1 + h_2} \Delta t; \end{aligned} \quad (8.17)$$

and at the outside surface

$$\begin{aligned} \frac{\Delta l - \Delta l_t}{l} &= (\alpha_1 - \alpha_2) \frac{2E_1 E_2 (h_1^3 h_2 + h_1^2 h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2} \Delta t = \\ &= (\alpha_1 - \alpha_2) \frac{h_1}{2(h_1 + h_2)} \Delta t. \end{aligned} \quad (8.18)$$

For the second metal
at the contact surface

$$\begin{aligned} \frac{\Delta l - \Delta l_t}{l} &= (\alpha_1 - \alpha_2) \frac{4E_1 E_2 (h_1 h_2^3 + h_1^2 h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2} \Delta t = \\ &= (\alpha_1 - \alpha_2) \frac{h_2}{h_1 + h_2} \Delta t; \end{aligned} \quad (8.19)$$

and at the outside surface

$$\begin{aligned} \frac{\Delta l - \Delta l_t}{l} &= -(\alpha_1 - \alpha_2) \frac{2E_1 E_2 (h_1 h_2^3 + h_1^2 h_2^2)}{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2} \Delta t = \\ &= -(\alpha_1 - \alpha_2) \frac{h_2}{2(h_1 + h_2)} \Delta t. \end{aligned} \quad (8.20)$$

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From eqs.(8.17), (8.18), (8.19), and (8.20) we see that in the case under examination (where $E_1 h_1^2 = E_2 h_2^2$), the "neutral" layers of both component metals, i.e., the layers where all tension is absent (in other words, the layers undergoing elongation specifically in accordance with the thermal elongation of the given metal), are found 2/3 of the distance away from the surface of contact through the given metal of the two in the strip.

Section 5. Stiffness of a Bimetal Strip

Now let us derive an equation describing the elasticity of a bimetal strip, i.e., let us determine the degree to which the strip is deflected by some external moment M . Such a formula may sometimes prove to be needed (see the Section on calculation of errors of friction).

Let us again derive an equation on the assumption that flat transverse sections of the bimetal strip normal to the surface of contact, remain flat and normal to the surface of the contact after the strip has been deflected by the moment M .

In the given case, the derivation is analogous to the derivation of the equation for deflections due to change in temperature. The only difference is that the equilibrium equations will not contain terms including $\alpha_1 \Delta t$ and $\alpha_2 \Delta t$. In addition, in the second equation (for moments), we replace zero on the right side by the active moment M or, in other words, we obtain the equations

$$\left. \begin{aligned} \int_0^{h_1} \frac{\Delta l + x\varphi}{l} E_1 b \, dx + \int_{-h_2}^0 \frac{\Delta l + x\varphi}{l_0} E_2 b \, dx &= 0; \\ \int_0^{h_1} \frac{\Delta l + x\varphi}{l} E_1 b x \, dx + \int_{-h_2}^0 \frac{\Delta l + x\varphi}{l} E_2 b x \, dx &= M, \end{aligned} \right\} \quad (8.21)$$

or

$$\begin{aligned} E_1 \left(\Delta l h_1 + \frac{h_1^2}{2} \varphi \right) + E_2 \left(\Delta l h_2 - \frac{h_2^2}{2} \varphi \right) &= 0; \\ \Delta l \left(\frac{h_1^2}{2} + \frac{h_1^3}{3} \varphi \right) + E_2 \left(-\Delta l \frac{h_2^2}{2} + \frac{h_2^3}{3} \varphi \right) &= \frac{Ml}{b}, \end{aligned} \quad \text{STAT}$$

from which it follows that

$$\left. \begin{aligned} \Delta l (E_1 h_1 + E_2 h_2) + \varphi \frac{E_1 h_1^2 - E_2 h_2^2}{2} &= 0; \\ \Delta l \frac{E_1 h_1^2 - E_2 h_2^2}{2} + \varphi \frac{E_1 h_1^3 + E_2 h_2^3}{3} &= \frac{Ml}{b}. \end{aligned} \right\} \quad (8.22)$$

Eliminating the unknown Δl from these two equations, we have

$$\varphi \left[\frac{(E_1 h_1^3 + E_2 h_2^3)(E_1 h_1 + E_2 h_2)}{3} - \frac{(E_1 h_1^2 - E_2 h_2^2)^2}{4} \right] = \frac{Ml(E_1 h_1 + E_2 h_2)}{b}.$$

After transformation on the left side of the equations, identical with those in Section 1, we have

$$\varphi \frac{4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 - (E_1 h_1^2 - E_2 h_2^2)^2}{12} = \frac{Ml(E_1 h_1 + E_2 h_2)}{b},$$

from which we derive

$$\varphi = Ml \frac{12(E_1 h_1 + E_2 h_2)}{b [4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2]} = k_2 Ml, \quad (8.23)$$

where the magnitude of k_2 is equal to

$$k_2 = \frac{12(E_1 h_1 + E_2 h_2)}{b [4E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2]}.$$

This is the reciprocal of the stiffness of the strip.

For the case of $E_1 h_1^2 = E_2 h_2^2$ we have

$$k_2 = \frac{3(E_1 h_1 + E_2 h_2)}{b E_1 E_2 h_1 h_2 (h_1 + h_2)^2}. \quad (8.24)$$

Thus, for example, if $E_1 = 2.16 \times 10^6$ kg/cm², $E_2 = 9.6 \times 10^5$ kg/cm², $h_1 =$ STAT = 0.4 mm, and $h_2 = 0.6$ mm (i.e., if the parameters of the strip are so chosen that $E_1 h_1^2 = E_2 h_2^2$) and $b = 4$ mm, we will have

$$k_2 = \frac{3(216 \cdot 10^4 \cdot 4 \cdot 10^{-2} + 96 \cdot 10^4 \cdot 6 \cdot 10^{-2})}{4 \cdot 10^{-1} \cdot 216 \cdot 10^4 \cdot 4 \cdot 10^{-2} \cdot 96 \cdot 10^4 \cdot 6 \cdot 10^{-2} (0,4 + 0,6)^2 10^{-2}} =$$

$$= \frac{3(864 + 576) 10^3}{4 \cdot 864 \cdot 576 \cdot 10} = \frac{10^2}{4608} \text{ kg}^{-1} \text{ cm}^{-2}.$$

Section 6. Thermal Deflection of the Strip in "Secondary" Compensation

In a number of mechanisms we still encounter bimetal strips operating as follows: The strip AB, fixed at one end A (in the same manner as in Fig.63), is also supported by a stop at some point C, between the A and B ends of the strip. In this design, we make use of the displacement of the free end B of the strip, as the temperature changes.

A design of this type is used, for example, for "secondary" thermal compensation in cases of so-called "kinematic" thermal compensation of an instrument. Here the use of this type of design permits adjustment of the intensity of this compensation, since the design permits displacement of the support along the strip, causing variations in the relationship between segments AC and CB of the strip. It is quite obvious that the smaller the $\frac{CB}{AC}$ ratio, the smaller will be the displacement of the B end of the strip with any change in temperature.

For example, in "secondary" temperature compensations, the object of which is to change the length of one of the arms of the multiplying mechanism of the instrument during variations in temperature (causing a change in the transmission ratio of the mechanism), this displacement of the stop is usually brought about as follows. The bimetal strip (1) (Fig.65) is riveted, at the point A, to a rigid pin (2) constituting the axis of a crank in the multiplying mechanism (the length of the crank in the given instance being the distance from the point B to the axis AA_1). The rod (2) contains a number of openings to receive the screw (3). This screw is inserted in one of these openings until its end rests against the bimetal strip (1). By transferring the screw (3) to another aperture, it is possible to vary the ratio

(1)

Technical drawing of a mechanical device, likely a probe or stylus. The device consists of a central vertical shaft with a pointed tip labeled A_1 and a top point labeled A . A curved arm is attached to the shaft, with a point labeled B at its end. A small component labeled 3 is attached to the side of the shaft. A point labeled C is marked on the curved arm. The shaft has a section labeled 2 near the top and a section labeled 1 near the curved arm. The curved arm is labeled D in the middle. The shaft has a section labeled A near the bottom. The shaft has a section labeled A_1 at the bottom tip.

Let us here apply a method analogous to that used in deriving the equation for deflection of a spring, supported at various points along its length. At the point C, the stop presses against the bimetal spring with a certain force Q. Thus the bimetal spring flexes under the action of two factors, one the change in temperature and the other the effect of the force Q. Let us deter-

mine the deflections f_B and f_C of the strip at the points B and C.

$$f_c = k_0 \frac{l_1^2}{2} \Delta t + Q k_2 \frac{l_1^3}{3},$$

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where l_1 is the length of the segment AC of the strip,

l_2 is the length of the segment CB.

Eliminating Q from these two equations, we obtain

$$\begin{aligned} f_B &= f_C \frac{2l_1 + 3l_2}{2l_1} + k_0 \left[\frac{(l_1 + l_2)^2}{2} - \frac{l_1(2l_1 + 3l_2)}{4} \right] \Delta t = \\ &= f_C \left(1 + \frac{3l_2}{2l_1} \right) + k_0 \left(\frac{l_2^2}{2} + \frac{l_1 l_2}{4} \right) \Delta t. \end{aligned} \quad (8.25)$$

The first term on the right side of this equation is independent of the temperature and depends only on the depth f_C , to which the stop screw (3) has been inserted. The second item is the variation in deflection of the point B, resulting from temperature fluctuations.

It is clear from the resultant expression that this thermal deflection does not depend on f_C , i.e., on the degree to which the screw (3) has been inserted, but only on the dimensions l_1 and l_2 of the segments of the bimetal strip and on the parameters of this strip.

In some devices, the following variation in the device under examination is encountered: Only some portion DB of the strip AB is bimetallic. Part AD is monometallic. For example, a solid monometal strip AB is taken, and another strip of a different metal is soldered to it along the segment DB. True, this arrangement is encountered mainly in old devices, but to the degree that such a variant is possible let us derive an expression for thermal deflection of the B end of a strip in this case.

The total deflection of the B end of the strip may also be expressed in the following way:

$$f_B = f_C + \varphi_C l_2 + k_0 \frac{l_2^2}{2} \Delta t, \quad (8.26) \quad \text{STAT}$$

where φ_C is the angle of variation in the direction of the tangent to point C, as compared to its direction in the initial position, when there has been neither vari-

ation in temperature nor in pressure Q , at the point C .

Let us designate the length of AD by l_3 , that of DC by l_1 , and that of CB by l_2 . Then the expression for the angle φ_C will have the form

$$\varphi_C = (\varphi_C)_t + (\varphi_C)_Q,$$

where $(\varphi_C)_t$ is the thermal deviation of the segment l_1 ,

$(\varphi_C)_Q$ is the deviation of the entire segment AC , under the effect of the load Q .

The deflection f_C at the point C can also be expressed as the sum of the two deviations

$$f_C = (f_C)_t + (f_C)_Q,$$

where $(f_C)_t$ is the thermal deviation at the point C ,

$(f_C)_Q$ is the deviation under the effect of the force,

$$(f_C)_Q = f_C - (f_C)_t, \\ (\varphi_C)_Q = (f_C)_Q \frac{(\varphi_C)_Q}{(f_C)_Q} = [f_C - (f_C)_t] \frac{(\varphi_C)_Q}{(f_C)_Q} = \left[f_C - k_0 \frac{l_1^2}{2} \Delta t \right] n,$$

from which we derive

$$\varphi_C = \left[f_C - k_0 \frac{l_1^2}{2} \Delta t \right] n + k_0 l_1 \Delta t, \quad (8.27)$$

where n is the magnitude of $\frac{(\varphi_C)_Q}{(f_C)_Q}$, which is not hard to determine.

If, in the segment AC of the strip, we take a segment of infinitesimal length dx along the distance x from the point C , then the force Q , in the given spring segment, will create a moment Q_x , so that this spring segment will be bent by an angle

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$$d\varphi = k_2 Q_x dx,$$

if it is in the bimetal portion of the strip DC, and by an angle

$$d\varphi = k_1 Q x dx,$$

if it is in the monometal segment AD of the strip.

Here $k_1 = \frac{1}{EJ}$, where E is the modulus of elasticity of the material in the monometal segment AD of the strip, and J is the moment of inertia of a cross section of the strip in this segment.

On this basis, the total $(\varphi_C)_Q$ angle of flexure of the segment AC of the strip is

$$(\varphi_C)_Q = Q \left[k_2 \int_0^{l_1} x dx + k_1 \int_{l_1}^{l_1+l_2} x dx \right] = Q \frac{k_2 l_1^2 + k_1 (l_2 + l_1)^2 - k_1 l_1^2}{2}.$$

The flexure $d\varphi$ of the elementary segment of the strip, dx in length, induces a displacement of the point C by $d(f_C)_Q$, equal to

$$d(f_C)_Q = x d\varphi = k_2 Q x^2 dx,$$

if the segment under consideration is in the bimetal portion of the strip DC, and by a magnitude of

$$d(f_C)_Q = x d\varphi = k_1 Q x^2 dx,$$

if the segment under consideration is in the monometal segment AD of the strip.

Therefore, the expression for the deflection $(f_C)_Q$ has the form

$$f_C = Q \left[k_2 \int_0^{l_1} x^2 dx + k_1 \int_{l_1}^{l_1+l_2} x^2 dx \right] = Q \frac{k_2 l_1^3 + k_1 (l_2 + l_1)^3 - k_1 l_1^3}{3}.$$

From this, we obtain

$$n = \frac{(\varphi_C)_Q}{(f_C)_Q} = \frac{3}{2} \frac{k_2 l_1^2 + k_1 (l_2 + l_1)^2 - k_1 l_1^2}{k_2 l_1^3 + k_1 (l_2 + l_1)^3 - k_1 l_1^3}. \quad (8.28)$$

Substituting expression (8.27) in eq.(8.26), we obtain

$$\begin{aligned} f_B &= f_C + l_2 \left[f_C - k_0 \frac{l_1^2}{2} \Delta t \right] n + k_0 l_1 l_2 \Delta t + k_0 \frac{l_2^2}{2} \Delta t = \\ &= f_C (1 + n l_2) + k_0 \left[\frac{l_2^2}{2} + l_1 l_2 \left(1 - \frac{n}{2} l_1 \right) \right] \Delta t. \end{aligned} \quad (8.29)$$

In the special case in which $l_3 = 0$, we have $n = \frac{3}{2l_1}$, and eq.(8.29) is transformed into eq.(8.25).

Section 7. Stiffness of Bimetal Strip in "Secondary" Temperature Compensation

As already indicated, the formula derived above for the rigidity of a bimetal strip (the value $\frac{1}{k_2}$) may be needed, for example, in calculating the errors of thermographs with bimetal pickup elements if there is some mechanical load on the bimetal (i.e., in calculating errors of friction).

However, in addition, this equation may also be necessary to confirm the fact that we have provided sufficiently rigid bimetal temperature compensation in the instrument. In point of fact, if the forces of friction in the multiplying mechanism are able to deflect the metal so that this may be observed by a change in the instrument readings, this will increase the error of friction in the instrument.

As already indicated, if the multiplying mechanism contains no flexible levers, then, in order to determine the errors of friction, all the forces and moments of friction are reduced to the pickup element, and we determine the variation in the magnitude to be measured, produced by the same deflection in the pickup as the deflection due to the reduced force (moment) of friction. But if the mechanism contains some flexible lever, then the error calculated in the above manner must be supplemented by a value $\frac{(F_{tp})_1}{c} \lambda$, where $(F_{tp})_1$ represents the force of friction, reduced to the flexible lever, of that portion of the mechanism between the given flexible lever and the pointer; c is the stiffness of the flexible lever when deflected in the direction of the reduced force of friction; and λ is a quantity indi-

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cating the ratio of the variation in the reading shown by the pointer (in units of the quantity being measured) to the deflection of the flexible lever.

Since, in the temperature compensation of instruments and in other devices, not only bimetal strips fixed at one end and free along their entire length are encountered but also designs in which the strip is supported at some point (see Fig.65), we derive a formula for the deflection of such a design under the effect of some force P , applied to the free end B of the strip.

Here, of course, as in the case of a spring supported at some point (see Chapter VI Sect.2), we have

$$f_B = P \frac{\partial f_B}{\partial P} + Q \frac{\partial f_B}{\partial Q};$$

$$f_C = P \frac{\partial f_C}{\partial P} + Q \frac{\partial f_C}{\partial Q},$$

where $\frac{\partial f_B}{\partial P}$, $\frac{\partial f_B}{\partial Q}$, $\frac{\partial f_C}{\partial P}$, and $\frac{\partial f_C}{\partial Q}$ are constants.

In a case in which the deflection f_C is constant, we have

$$df_B = dP \frac{\partial f_B}{\partial P} + dQ \frac{\partial f_B}{\partial Q};$$

$$0 = dP \frac{\partial f_C}{\partial P} + dQ \frac{\partial f_C}{\partial Q}.$$

From this, eliminating dQ , we obtain

$$\frac{df_B}{dP} = \frac{\frac{\partial f_B}{\partial P} \frac{\partial f_C}{\partial Q} - \frac{\partial f_B}{\partial Q} \frac{\partial f_C}{\partial P}}{\frac{\partial f_C}{\partial Q}}, \quad (8.30)$$

while, as shown above (in calculating the deflection of a spring supported at a single point), we always have

$$\frac{\partial f_B}{\partial Q} = \frac{\partial f_C}{\partial P}.$$

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If the strip AB is of the bimetal type, the expressions $\frac{\partial f_B}{\partial P}$, $\frac{\partial f_B}{\partial Q}$, $\frac{\partial f_C}{\partial P}$, and $\frac{\partial f_C}{\partial Q}$ differ from the same expressions for springs supported at a single point only in that here, instead of the factor $\frac{1}{2J}$ we find the factor k^2 , and therefore we will have

$$\begin{aligned} \frac{df_B}{dP} &= k_2 \frac{\frac{(l_1+l_2)^3}{3} - \frac{l_1^3}{3} - \left| \frac{l_1^2(2l_1+3l_2)}{6} \right|^2}{\frac{l_1^3}{3}} = \\ &= k_2 \left[\frac{(l_1+l_2)^3}{3} - \frac{l_1(2l_1+3l_2)^2}{12} \right] = k_2 \frac{l_1^2(3l_1-l_2)}{12}. \end{aligned} \quad (8.31)$$

In a case in which the segment AD of the strip is monometallic, we will have more complex expressions for the quantities

$$\begin{aligned} &\frac{\partial f_B}{\partial P}, \quad \frac{\partial f_B}{\partial Q}, \quad \frac{\partial f_C}{\partial P} \text{ и } \frac{\partial f_C}{\partial Q}: \\ \left. \begin{aligned} \frac{\partial f_B}{\partial P} &= k_1 \left[\frac{(l_3+l_1+l_2)^3}{3} - \frac{(l_1+l_2)^3}{3} \right] + k_2 \frac{(l_1+l_2)^3}{3}; \\ \frac{\partial f_B}{\partial Q} &= \frac{\partial f_C}{\partial P} = k_1 \left[\frac{(l_3+l_1)^3}{3} - \frac{l_1^3}{3} + \frac{(l_2+l_1)^2 l_2}{2} - \frac{l_1^2 l_2}{2} \right] + \\ &\quad + k_2 \left(\frac{l_1^3}{3} + \frac{l_1^2 l_2}{2} \right); \\ \frac{\partial f_C}{\partial Q} &= k_1 \left[\frac{(l_3+l_1)^3}{3} - \frac{l_1^3}{3} \right] + k_2 \frac{l_1^3}{3}. \end{aligned} \right\} \quad (8.32) \end{aligned}$$

Substituting these expressions in eq.(8.30), we obtain

$$\begin{aligned} \frac{df_B}{dP} &= k_1 \left[\frac{(l_3+l_1+l_2)^3}{3} - \frac{(l_1+l_2)^3}{3} \right] + k_2 \frac{(l_1+l_2)^3}{3} - \\ &\quad \left\{ k_1 \left[\frac{(l_3+l_1)^3}{3} - \frac{l_1^3}{3} \right] - k_2 \frac{l_1^3}{3} + k_1 \left[\frac{l_2(l_3+l_1)^2}{2} - \frac{l_2 l_1^2}{2} \right] + k_2 \frac{l_2 l_1^2}{2} \right\} = \\ &\quad \frac{k_1 \left[\frac{(l_3+l_1)^3}{3} - \frac{l_1^3}{3} \right] + k_2 \frac{l_1^3}{3}}{k_1 \left[\frac{(l_3+l_1+l_2)^3}{3} - \frac{(l_1+l_2)^3}{3} \right] + k_2 \frac{(l_1+l_2)^3}{3}} \end{aligned}$$

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$$\begin{aligned}
&= k_1 \left[\frac{(l_3 + l_1 + l_2)^3}{3} - \frac{(l_1 + l_2)^3}{3} \right] + k_2 \frac{(l_1 + l_2)^3}{3} - k_1 \left[\frac{(l_3 + l_1)^3}{3} - \frac{l_1^3}{3} \right] - \\
&\quad - k_2 \frac{l_1^3}{3} - l_2 [k_1 (l_3 + l_1)^2 - k_1 l_1^2 + k_2 l_1^2] - \\
&\quad - \frac{3l_2^3 [k_1 (l_3 + l_1)^2 - k_1 l_1^2 + k_2 l_1^2]^2}{4 [k_1 (l_3 + l_1)^3 - k_1 l_1^3 + k_2 l_1^3]} = k_1 \frac{(l_3 + l_1 + l_2)^3 - (l_1 + l_2)^3 - (l_3 + l_1)^3 + l_1^3}{3} + \\
&\quad + k_2 \frac{(l_1 + l_2)^3 - l_1^3}{3} - l_2 [k_1 (l_3 + l_1)^2 - k_1 l_1^2 + k_2 l_1^2] \left(1 + \frac{1}{2} l_2 n \right), \quad (8.33)
\end{aligned}$$

where n is the magnitude indicated in eq.(8.28).

In the special case in which $l_3 = 0$, all the terms of the expression containing the factor k_1 are eliminated, and the equation that remains reads

$$\frac{df_B}{dP} = k_2 \left[\frac{(l_1 + l_2)^3}{3} - \frac{l_1^3}{3} - l_2 l_1^2 - \frac{3}{4} l_2^2 l_1 \right] = \frac{k_2 l_2^2 (3l_1 + 4l_2)}{12},$$

or, in other words, eq.(8.33) is converted into eq.(8.31).

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CHAPTER I.1

TRANSMITTING MECHANISMS FOR INSTRUMENTS

Section 1. General Purpose and Major Types of Components of Transmitting Mechanisms in Measuring Instruments1. The Basic Purpose of Transmitting Mechanisms

The transmitting mechanisms of measuring instruments fulfill three major functions:

1. The transmitting mechanism transmits the motion of a specific point in the pickup to a point convenient for reading or recording. In actuality, the pickup, due to its special characteristics or other considerations of design, may be so located as to interfere with installation of a dial in a convenient place; in this case, the deflection of the pointer is performed directly by the pickup element.

2. The transmitting/amplifying element serves to increase the rate of deflection of the instrument pointer, i.e., the end of the pointer undergoes considerably greater displacement than does the particular point of the pickup whose motion is transmitted to the pointer. This increases the sensitivity of the instrument.

Here it should be noted that the displacements in most pickups are exceedingly small.

3. Transmitting mechanisms are often used to vary the nature of the instrument dial, i.e., to cause the ratio between the dial graduations to differ from the ratio of the corresponding displacements of the pickup point which actuates the transmitting mechanism.

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This is done in cases in which the dial, for some reason, is not graduated in the manner that would result if the transmission ratio were constant at all readings of the instrument. Thus, in certain instruments an equi-graduated dial is required, although the displacement of the corresponding point of the pickup is not proportional to the quantity being measured.

Variation in the transmission ratio of the mechanism as a whole may be attained by providing individual elements of the mechanisms with variable transmission ratios, i.e., with variable ratios of the rates of displacement of the point of contact with the following component and the point of contact with the preceding component.

This property of transmitting mechanisms is made use of, as are certain others, in the calibration of instruments.

2. Major Types of Transmitting Connections

The majority of the most widely-used types of transmitting mechanisms used in instrument manufacture have variable transmission ratios.

The fact is that, aside from gear-type transmissions, there is virtually no transmitting mechanism assembly with a constant transmission ratio. Most instrument transmitting mechanisms consist of combinations of levers connected by links or attached to each other directly. In such cases, the transmission ratio depends on the angles between the levers, between the levers and links, and in general between the individual parts of the mechanism.

The following are the types of transmissions most frequently encountered in the transmitting mechanisms of measuring instruments:

1. Transmission by articulated joints;
2. Transmission by sliding connections;
3. Chain transmission;
4. Gear transmission.

Let us examine some of the types of transmissions most frequently encountered.

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3. Flat-Hinge Joints

Simple Crank Mechanisms and Offset Crank Mechanism. In articulated joints, the motion of one part of the mechanism is transmitted to another by means of a link connected by pins to specific points of the parts in question. Of the various types of transmitting mechanisms in this category, the crank is the type most commonly employed (Fig.66).

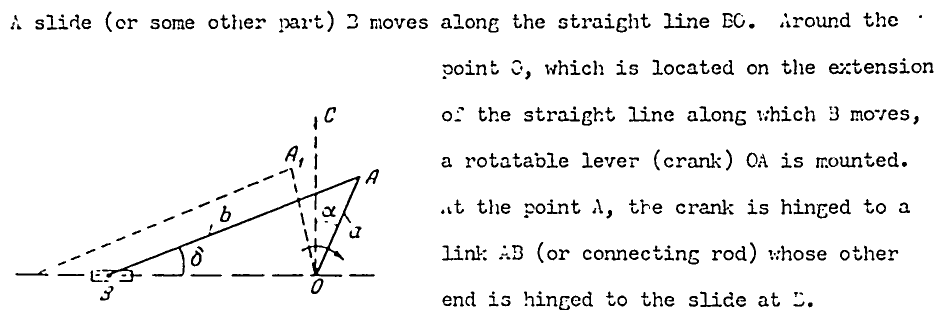


Fig.66 - Schematic Drawing of
Simple Crank Mechanism

If we denote by α the angle between the crank OA and the normal to the straight line BC , while δ is the angle between the connecting rod AB and the straight line BC , a is the length of the lever OA and b the length of the connecting rod AB , the degree of displacement x of the slide from its initial position can be given as

$$x = OB_0 - OB = a (\sin \alpha - \sin \alpha_0) + b \left[\sqrt{1 - \left(\frac{a}{b}\right)^2 \cos^2 \alpha_0} - \sqrt{1 - \left(\frac{a}{b}\right)^2 \cos^2 \alpha} \right] \quad (9.1) \quad \text{STAT}$$

where OB_0 and α_0 are, respectively, the magnitudes of OB and α when the mechanism is in its initial position.

In this equation, the angle α is calculated from the line OC in the direction indicated by the arrow. Thus, the angles α and α_0 may be either positive or negative (in the position OA_1 , shown by a broken line, the angle α_1 is negative). We

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will consider the quantity x as positive when the slide moves from left to right.

The ratio of the rate of displacement of the slide to the rate of rotation of the crank is the derivative of x relative to α

$$\begin{aligned} \frac{dx}{d\alpha} &= a \cos \alpha - b \frac{\frac{a^2}{b^2} \cos \alpha \sin \alpha}{\sqrt{1 - \left(\frac{a}{b}\right)^2 \cos^2 \alpha}} = \\ &= a \left[\cos \alpha - \frac{a}{b} \frac{\sin \alpha \cos \alpha}{\sqrt{1 - \left(\frac{a}{b}\right)^2 \cos^2 \alpha}} \right]. \end{aligned} \quad (9.2)$$

The following symbols will be used in our further discussion.

If, at any of the joints between the parts of the mechanism already referred to, or at any other joint, the part m transmits motion to the part n (reckoning the transmission from the pickup to the arrow, rather than vice versa), then that portion of the part m which transmits motion to the next part of the mechanism will be denoted as the "driving portion" of the part m , while the portion of the part n , which directly receives the motion of the preceding part m , or the motion of the link connecting it to the part m , will be designated the "driven portion" of the part n .

For example, if the rotation of the crank OA (see Fig. 66) is connected in some way with a displacement of the pickup (directly or by means of some transmitting mechanism), the rotation of the crank causes the slide to move, while the motion of the slide is transferred, by a further component of the transmitting mechanism, to the pointer; in this case we will refer to the slide as driving and to the crank as driven.

In the case of a driving crank and a driven slide, the overall transmission ratio of the instrument (from the pickup to the pointer) will contain a factor $\frac{dx}{d\alpha}$, calculated from eq. (9.2). However, if the slide is driving and the crank driven, the transmission ratio will include a factor of reciprocal magnitude.

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A crank mechanism is called offset if the continuation of the straight line BD along which the slide is displaced, does not pass through the axis of rotation of the crank (Fig.67).

We know that, if we denote by d the distance OO_1 from the point O to line BD , the displacement x of the slide from its initial position may be written in the form

$$x = O_1B_0 - O_1B = a(\sin \alpha - \sin \alpha_0) + b \left[\sqrt{1 - \left(\frac{a \cos \alpha_0 - d}{b}\right)^2} - \sqrt{1 - \left(\frac{a \cos \alpha - d}{b}\right)^2} \right]. \quad (9.3)$$

The ratio of the rate of displacement of the slide to the angular rate of rotation of the crank will be

$$\begin{aligned} \frac{dx}{d\alpha} &= a \cos \alpha - b \frac{a \sin \alpha (a \cos \alpha - d)}{b^2 \sqrt{1 - \left(\frac{a \cos \alpha - d}{b}\right)^2}} = \\ &= a \left[\cos \alpha - \frac{(a \cos \alpha - d) \sin \alpha}{b \sqrt{1 - \left(\frac{a \cos \alpha - d}{b}\right)^2}} \right]. \end{aligned} \quad (9.4)$$

We know that, if the ratio between the lengths of the rod b and the crank a is large enough, a simplification of eqs.(9.1), (9.2), (9.3), and (9.4) is often permissible. Laying out the value $\sqrt{1 - \left(\frac{a}{b}\right)^2 \cos^2 \alpha}$ of eq.(9.1) and the quantity $\sqrt{1 - \left(\frac{a \cos \alpha - d}{b}\right)^2}$ of eq.(9.3) in series, in accordance with the angle α , and ignoring the third and subsequent terms in these series, in view of the small values thereof, we obtain the following formulas:

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For an ordinary crank mechanism

$$x = a \left[(\sin \alpha - \sin \alpha_0) + \frac{a}{4b} (\cos 2\alpha - \cos 2\alpha_0) \right]; \quad (9.5)$$

For an offset crank mechanism

$$x = a \left[(\sin z - \sin z_0) + \frac{a}{4b} (\cos 2z - \cos 2z_0) - \frac{d}{b} (\cos z - \cos z_0) \right]. \quad (9.6)$$

In accordance with this, we obtain in place of eq.(9.2), the following equation

$$\frac{dx}{dz} \approx a \left(\cos \alpha - \frac{a}{2b} \sin 2\alpha \right) \quad (9.7)$$

and in place of eq.(9.4), the equation

$$\frac{dx}{dz} \approx a \left(\cos \alpha - \frac{a}{2b} \sin 2\alpha + \frac{d}{b} \sin \alpha \right). \quad (9.8)$$

Some sources state that simplifications such as these may be used [i.e., that eqs.(9.5), (9.6), (9.7), and (9.8) may be used instead of (9.1), (9.2), (9.3), and (9.4)] in case the $\frac{b}{a}$ ratio is not less than 3.

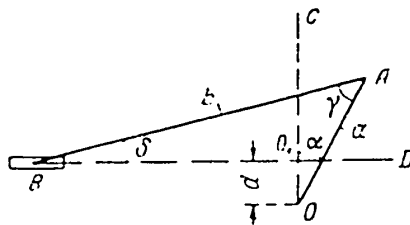


Fig.67 - Schematic Drawing

Offset Crank Mechanism

For purposes of calculating the transmission mechanisms of instruments where high precision is required, this specification can be made somewhat stricter. It may be recommended that we do not use the

approximate eqs.(9.5), (9.6), (9.7), and (9.8) in cases in which the $\frac{b}{a}$ ratio is less than 3.5 or 4.

If the ratio of b to a is very large (i.e., 7 or more), and if, in this connection, the magnitude of d is such that angle δ at various positions of the mechanism remains nearly zero (i.e., that the quantity d is positive and somewhat smaller than a , which is the most common situation with such mechanisms), then it may sometimes be possible to make use of even more simplified equations

$$x \approx a (\sin \alpha - \sin \alpha_0) \quad (9.9)$$

and

$$\frac{dx}{d\alpha} \approx a \cos \alpha.$$

Actually, if, without varying the quantities a and d , we lengthen the link b , the average absolute magnitude of the angle δ will decline by approximately as large a factor as that by which the link b is increased. The change $\Delta\delta$ in the angle δ for an identical rotation of the crank, will decrease by the same factor.

Based on the theorem of finite differences, we have

$$b (\cos \delta_1 - \cos \delta_{11}) = -b (\delta_1 - \delta_{11}) \cdot \sin \delta_{cp} \quad b \Delta \delta \sin \delta_{cp},$$

where δ_{cp} is a mean between δ_1 and δ_{11} .

As the length of the link b increases, the magnitude of $\Delta\delta$ declines by approximately the same factor. Therefore, the product $b \cdot \Delta\delta$ undergoes virtually no change in magnitude. However, the factor $\sin \delta_{cp}$ is usually smaller when the link b is larger; therefore, if b is large enough (relative to a), we may ignore the term $b(\cos \delta_0 - \cos \delta)$ in eqs.(9.1) and (9.3), i.e., the term containing a factor in the brackets.

It should be noted that this simplified formula may also be used in a case in which the ratio of b to a is small, but in this case the entire range of rotation of the crank is small, and the angle α is at all times very close to zero, particularly if the link b is nearly perpendicular to the crank a , i.e., if the angle δ is small.

In this case, the variation $\Delta\delta$ in the angle δ will be no more than a fraction of the variation $\Delta\alpha$ in the angle α . In point of fact, Figs.66 and 67 make it clear that

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$$\Delta(b \sin \delta) = \Delta(a \cos \alpha),$$

but

$$\Delta(a \cos \alpha) = -a \Delta \alpha \sin \alpha_{cp},$$

where α_{cp} is a mean value of the angle α in the range of variation $\Delta \alpha$.

For example, if in the total range of variation in the angle α , this angle is zero when the mechanism is in the same position, and the maximum absolute magnitude of the angle α is equal to α_1 and small, we may then write, in first approximation

$$[\Delta(a \cos \alpha)]_{\max} \approx \mp a \alpha_1 \alpha_{cp},$$

and therefore

$$[\Delta(\sin \delta)]_{\max} \approx \mp \frac{a}{b} \alpha_1 \alpha_{cp}.$$

if the angles δ are small, then

$$(\Delta \delta)_{\max} \approx \mp \frac{a}{b} \alpha_1 \alpha_{cp},$$

and

$$[\Delta(b \cos \delta)]_{\max} \approx -b (\Delta \delta)_{\max} \sin \delta_{cp} = \pm a \alpha_1 \alpha_{cp} \sin \delta_{cp}.$$

The magnitude of the first term in eq. (9.3) when the angles α are small, is approximately equal to

$$a(\sin \alpha - \sin \alpha_0) \approx a(\alpha - \alpha_0) = a \Delta \alpha,$$

and its maximum magnitude is $a(\alpha_1)_{\max}$.

If the angles α are small enough, this value is many times larger than $a \alpha_1 \alpha_{cp} \sin \delta_{cp}$, since the quantity $a \alpha_1$, which by itself is smaller than $a(\alpha_1)_{\max}$, is

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further multiplied by the two small terms α_{cp} and $\sin \delta_{cp}$.

In order to make practical use of eqs.(9.2) and (9.4) in the calibration of instruments (when it is necessary to determine, approximately, by eye, the manner in which the position of the mechanism must be changed) the preferable equation for ready reference is that in which the value of $\frac{dx}{d\alpha}$ is expressed as an implicit function of α , so that eqs.(9.2) and (9.4) can be presented in the form

$$\frac{dx}{d\alpha} = a \left(\cos \alpha - \frac{\sin \alpha \sin \delta}{\cos \delta} \right) = \frac{a \cos (\alpha + \delta)}{\cos \delta}.$$

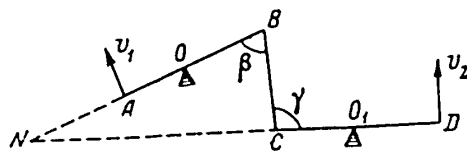
However, since $\alpha + \delta = \frac{\pi}{2} - \gamma$, where γ is the angle between the crank and the link, we have

$$\frac{dx}{d\alpha} = \frac{a \sin \gamma}{\cos \delta}. \quad (9.10)$$

Since the magnitude of $\cos \delta$ varies little compared to the variation in $\sin \gamma$, the calibrator, in changing the angle between the crank and the link, must determine how this will affect the transmission ratio of the mechanism.

Conclusions Valid for All Flat Hinge Joints. Expressions analogous to those of eq.(9.10) can be obtained for any flat hinge mechanism.

For example, let us deal with a mechanism (Fig.68) in which a lever AB, rotating about the axis O, is articulated to



the link BC and the point B, the other end of which is hinged at the point C to the lever CD rotating about the axis O_1 .

Fig.68 - Flat Hinge Joint Mechanism

Since the velocity of motion of the point B is normal to the lever OB, and the

velocity of motion of the point C is normal to the lever O_1C , the instantaneous center of rotation of the link BC, when the mechanism is in a given position, lies at the point N, forming the intersection of the extensions of the levers OB and O_1C ; the velocities v_3 and v_4 of motion of the points B and C, when the mechanism is in a

given position, have the following relationship

$$\frac{v_4}{v_3} = \frac{NC}{NB}$$

The triangle BNC yields

$$\frac{NC}{NB} = \frac{\sin \beta}{\sin \gamma}$$

Further,

$$v_3 = OB \cdot \omega_1;$$

$$v_4 = O_1C \cdot \omega_2,$$

where ω_1 is the velocity of rotation of the lever AB;

ω_2 is the velocity of rotation of the lever CD.

Thus,

$$\frac{O_1C\omega_2}{OB\omega_1} = \frac{\sin \beta}{\sin \gamma},$$

from which

$$\frac{\omega_2}{\omega_1} = \frac{OB \sin \beta}{O_1C \sin \gamma}.$$

Let us determine the ratio of the velocity v_D of motion of the point D to the velocity v_A of motion of the point A.

We have

$$\frac{v_3}{v_4} = \frac{O_1D}{O_1C}; \quad \frac{v_3}{v_1} = \frac{OB}{OA} \text{ and } \frac{v_4}{v_2} = \frac{\sin \beta}{\sin \gamma}.$$

Multiplication of these three equations yields

$$\frac{v_2}{v_1} = \frac{O_1D \cdot OB \sin \beta}{OA \cdot O_1C \sin \gamma} \quad (9.11)$$

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By this method, i.e., by the method of instantaneous center points, it would also be easy to obtain eq.(9.10). This method is generally applicable to a very large variety of transmission linkages. To solve analogous problems another method applicable to all types of linkages - hinged or sliding - both for plane and three-dimensional systems, may find even wider application.

This method is based on the fact that the work of two forces applied to the two ends of a transmission, must be equal if we disregard inertia and friction (i.e., if we think in terms of an ideal system). From this, for a given type of transmission (see Fig.68 for example), the velocity of motion of the points A and D (the initial and terminal points in the transmission), on displacement of the point A, will be inversely proportional to the forces that must be applied to the points A and D (in the direction of motion of these points) in order for the entire system to remain in equilibrium.

Often this method may significantly simplify the derivation of the equation for a transmission ratio, since in many cases it is easier to grasp the manner in which various changes in the relative position of parts of a transmitting mechanism affect the ratio between the balancing forces than it is to make a direct analysis of the influence of such changes on the ratio between the velocities of various points in the mechanism.

For example, in the case depicted in Fig.68, simple applications of statics will yield

$$\frac{Q}{P} = \frac{OA \cdot O_1C \sin \gamma}{O_1D \cdot OB \sin \beta}, \quad (9.11a)$$

where P and Q are the forces that must be applied to the points A and D, respectively, in the direction of their motion, in order for the system to be in equilibrium.

Equation (9.11) is obtained from this last equation.

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4. Transmissions with Sliding Joints

A transmission is termed sliding if one moving part causes motion in another with which it is in contact but to which it is not hinged by articulation. Constant contact between these parts is effected by a spring of some sort (i.e., by the hair-spring of the instrument). The following definition is also applicable to most of the conventional types of sliding connections: In such a joint, a lever must at all times be in contact, along its side or end, with a cam either in linear or rotating motion.

Should the cam be of the rotating variety, its axis of rotation may be either parallel or normal to the axis of rotation of the lever in contact therewith. Cams may be of any shape, but instruments in mass and series production almost never use cams of regular shape. As previously indicated, the reason for this is that the

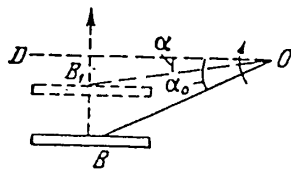


Fig. 69 - Sinusoidal Cam Joint:

Diagram

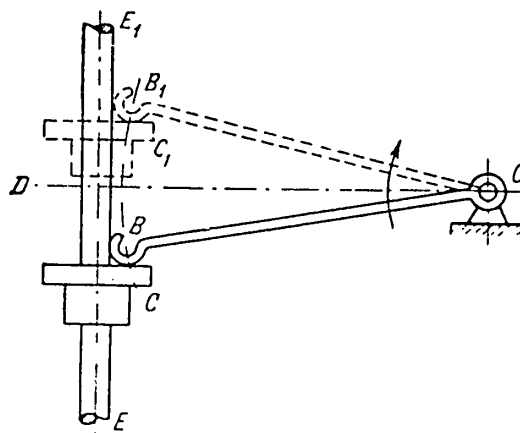


Fig. 70 - Sinusoidal Cam Joint

parts going into such mechanisms (particularly the pickup elements) are not ^{completely} identical, so that the shape of a cam that is satisfactory for one instrument may prove unsatisfactory for another. This being the case, it is usually only the following types of simple cams that are used in instruments.

Type I. Here, the transmission is one in which the profile of a cam in linear and advancing motion, in contact with the end of the lever OB, constitutes a plane normal to the direction of motion of the cam (Fig.69).

In this case, if x designates the displacement of the cam, and α the angle between the lever and the straight line OD, normal to the direction of motion of the cam, we will have

$$x = a (\sin \tau - \sin \alpha_0), \quad (9.12)$$

where a is the length of the lever OB;

α_0 is the original magnitude of the angle α .

We will consider the angle α and the quantity x to be positive in the directions indicated by arrows (therefore, in Fig.69, the angles τ and α_0 are both negative).

If the end of a lever has a curvature so large that it cannot be ignored (Fig.70), then the angles α must be read between the lines OD and CB, connecting the center of curvature B with the axis of rotation of the lever O, and not between the lines OD and CC, where C is the point of contact between the lever and the plane of the cam.

In fact, if the periphery of curvature is a circle, the projection of the line BP₁ on the direction of motion of the cam is obviously equal to the displacement of the cam itself.

Equation (9.12) yields

$$\frac{dx}{d\alpha} = a \cos \alpha. \quad (9.13)$$

In the case in which the cam is the driving element and the lever OB the driven element, a factor of the reciprocal magnitude enters the general transmission ratio of the instrument mechanism.

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Type II. This is a transmission in which, with advancing linear motion of the

cam, the lever OB is in all positions in contact with the same point of the cam (Fig.71). This latter condition is realized either by means of a slot fastened (normal to the lever OB) to the part with which it is in contact (parallel to the axis of the lever), or by means of some kind of point fastened to the part in advancing

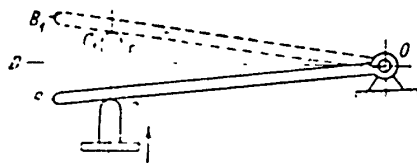


Fig.71 - Tangential Cam Coupling

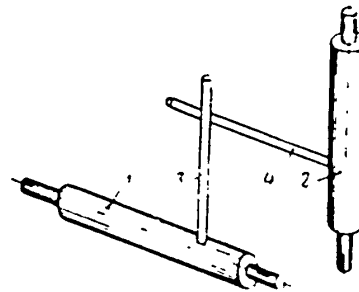


Fig.72 - Dog Transmission

motion and in contact with some bar rotating about the axis C (in the latter case, the bar will be the lever OB).

In this case, we obtain

$$x = a(\operatorname{tg} \alpha - \operatorname{tg} \alpha_0). \quad (9.14)$$

where x is the displacement of the cam (at the point C);

α is the angle between the straight line OD, normal to the motion of the cam, and the lever OB;

α_0 is the initial magnitude of the angle α ;

a is the distance between the angle of rotation O and the trajectory of the point of contact C of the cam, with the lever OB, i.e., $a = OE$.

From this, we derive

$$\frac{dx}{d\alpha} = \frac{a}{\cos^2 \alpha}. \quad (9.15) \quad \text{STAT}$$

Type III. Dog transmission (Fig.72).

On rotation of the roller (1), the dog (3) is forced against the dog (4) and

compels the roller (2) to rotate. If the dog (3) is normal to the roller (1) and the dog (4) is normal to the roller (2), then the dog (3) is shifted into a plane normal to the axis (1), and the dog (4) into a plane normal to the axis (2).

Since the intersection of the two planes is a straight line, the geometric locus of the points of contact of the dogs (3) and (4) is a straight line normal to the axes (1) and (2). Thus we have, as it were, a combination of two linkages corresponding to the second case; in one of these, the lever is driving (dog 3), while in the other it is driven. The role of the cam for the two levers is played by their point of contact (or, to be more exact, each of the dogs serves as a cam for the other).

In the simplest case, when the angles of the rollers (1) and (2) are in a single plane, we have

$$a \operatorname{tg} \alpha = b \cdot \operatorname{tg} \beta. \quad (9.16)$$

In this equation, α is the angle of rotation of the roller (1) from the position in which the dog (3) is parallel to the dog (2), i.e., lies in a plane corresponding to the center position of this transmission joint; β is the angle of rotation of the roller (2) from its center position; a is the distance of the axis (1) from the straight line, indicating the geographic locus of the points of contact of the dogs; and b is the distance of the axis (2) from this straight line.

Thus,

$$\beta = \operatorname{arc} \operatorname{tg} \left(\frac{a}{b} \operatorname{tg} \alpha \right),$$

from which

$$\frac{d\beta}{d\alpha} = \frac{a}{b} \frac{\cos^2 \alpha}{1 + \left(\frac{a}{b}\right)^2 \operatorname{tg}^2 \alpha} = \frac{a}{b} \frac{1}{\cos^2 \alpha + \left(\frac{a}{b}\right)^2 \sin^2 \alpha}. \quad (9.17) \quad \text{STAT}$$

Often, in calibration, the dogs are bent so that the end of one (and sometimes

of both) is removed from a position normal to its axis of rotation. This is usually done to only one of the dogs, and customarily to the leader. In this case, the end of the dog (3), contacting the dog (4), describes not a plane but a conical surface, so that the geometric locus of the points of contact of the dogs forms a hyperbola

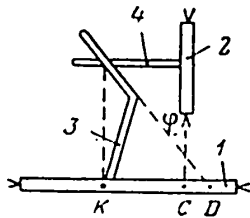


Fig. 73 - Calibration of Dog Transmission by Bending the Driving Dog

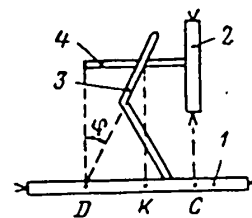


Fig. 74 - Another Case of Calibration of Dog Transmission by Bending the Driving Dog

(Figs. 73 and 74).

Let us analyze a case in which the dog (3) has been bent in the plane in which the axis of the roller (1) is located. Let us consciously extend the straight line along which the end of the roller (3) is located, prior to intersection with the axis of the roller (1) at the point D. Let the continuation of the axis (2) intersect the axis (1) and the point C. Let us designate by b the distance from the point C to the point D, taking as positive the direction from the point C toward the projection K (on the axis 1) of the point of contact of the dogs.

Thus, in Fig. 73 the value of b is negative. The angle of slope of the end of the dog (3) is denoted by ϕ , and we will take ϕ to be positive when the end of the dog (3) is bent in a direction opposite to that of the roller (2), and negative when it is facing the roller (2) (see Fig. 74). The distance of the axis (1) from the plane in which the dog (4) is in motion will, as before, be denoted by a . STAT

Let the roller (1) rotate from its center position through an angle τ . Let us

depict the displacement of the point of contact of the dog (4) and the dog (3) in the direction in which it is shifted (Fig.75).

Let E be the point of contact of the dogs (3) and (4) and let the projection of this point on a straight line OL, which is the direction of the dog (4), with the mechanism in center position. Then the distance OE will be $a \tan \gamma$ [a relationship obtained by intentionally dropping a perpendicular from the point L on the axis of the roller (1)].

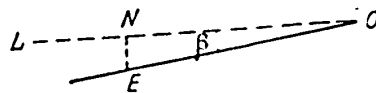


Fig.75 - See Derivation of Formula
for Dog Transmission when Driving
Dog is Bent

Further, since the perpendicular from the point E onto the axis (1) is $\frac{a}{\cos \alpha}$, the distance DK from the point D (see Figs.73 and 74) to the base of the perpendicular in question is $\frac{a}{\cos \alpha} \tan \gamma$.

Consequently, the magnitude of CK (see Fig.75) is

$$b + \frac{a}{\cos \alpha} \tan \gamma.$$

Thus,

$$\tan \beta = \frac{a \tan \alpha}{b + \frac{a}{\cos \alpha} \tan \gamma},$$

or

$$\beta = \arctan \left[\frac{a \tan \alpha}{b + \frac{a}{\cos \alpha} \tan \gamma} \right], \quad (9.12)$$

and, consequently,

$$\frac{d\beta}{d\alpha} = \frac{\frac{a}{\cos^2 \alpha} \left(b + \frac{a}{\cos \alpha} \tan \gamma \right) - a \tan \alpha \frac{a \sin \alpha}{\cos^2 \alpha} \tan \gamma}{1 + \left(\frac{a \tan \alpha}{b + \frac{a}{\cos \alpha} \tan \gamma} \right)^2} =$$

STAT

$$= \frac{\frac{a}{\cos^2 \alpha} \left(b + \frac{a}{\cos \alpha} \operatorname{tg} \varphi - a \frac{\sin^2 \alpha}{\cos \alpha} \operatorname{tg} \varphi \right)}{\left(b + \frac{a}{\cos \alpha} \operatorname{tg} \varphi \right)^2 + (a \operatorname{tg} \alpha)^2} = \frac{a (b + a \cos \alpha \operatorname{tg} \varphi)}{(b \cos \alpha + a \operatorname{tg} \varphi)^2 + (a \sin \alpha)^2} \quad (9.19)$$

Cases in which the ends of both dogs are bent are rare.

Type IV. The so-called rocker-arm transmission differs from the first variety of simple sliding linkages in that a cam with a flat surface, in contact with the end of the lever AB (Fig. 76a), moves not in linear fashion - its surface does not remain parallel to its initial position - but rocks relative to an axis O_1 .

Let us examine only the simplest case, in which the axis O_1 of rotation of the plane of the cam lies in that plane itself.

If the axis O_1 of rotation of the cam O_1CD is, furthermore, parallel to the axis O of rotation of the lever AB (see Fig. 7(a)), it may be stated that the transmission in question also differs from the second variety of cam drives discussed above (see Type II) only in that the point B, at which the cam is in contact with the lever OB in Fig. 7(a), is displaced not linearly but along a curve. In this case, the cam O_1C_1D may also take the form of a lever rather than of a plane (Fig. 76b); however, in this case the lever AB must terminate in a dog of some type DD_1 , parallel to the axes O and O_1 . In this type of design, either of the two levers can be considered the cam.

When the axes O and O_1 are parallel, we have the following relationship (see Fig. 76b):

$$\operatorname{tg} \beta = \frac{BD}{O_1D} = \frac{a \sin \alpha}{c - a \cos \alpha} \quad (9.20)$$

where β is the angle between the plane C_1D (or the lever O_1D) and the plane OO_1 ;

α is the angle between the lever OB and the plane OO_1 ;

a is the length of the lever OB;

c is the distance between the axes O and O_1 .

STAT

From this we find that

$$\beta = \arctg \left(\frac{a \sin \alpha}{c - a \cos \alpha} \right);$$

$$\frac{d\beta}{d\alpha} = \frac{\frac{a \cos \alpha (c - a \cos \alpha) - a^2 \sin^2 \alpha}{(c - a \cos \alpha)^2}}{1 - \frac{a^2 \sin^2 \alpha}{(c - a \cos \alpha)^2}} = \frac{a (c \cos \alpha - a)}{c^2 + a^2 - 2ac \cos \alpha} \quad (9.21)$$

which is also obvious from the fact that

$$\frac{d\beta}{d\alpha} = \frac{a}{O_1 B} \cos \gamma,$$

in which

$$O_1 B = \sqrt{c^2 + a^2 - 2ac \cos \alpha};$$

$$\cos \gamma = \frac{BE}{O_1 B} = \frac{c \cos \alpha - a}{\sqrt{c^2 + a^2 - 2ac \cos \alpha}}.$$

If the axes O and O_1 are normal to each other (Fig. 77), then the term $a \cos \alpha$ in the denominator of eq. (9.20) is eliminated; here, the letter c denotes the distance from the axis O_1 to the plane into which the B end of the lever CD is shifted. In this case, we obtain:

For the case in which the axes O and O_1 lie in the same plane

$$\operatorname{tg} \beta = \frac{a \sin \alpha}{c} \quad (9.22)$$

and for the case in which the axes O and O_1 are not in a single plane

$$\operatorname{tg} \beta = \frac{a \sin \alpha \pm d}{c} \quad (9.23)$$

In this case, the angle β is measured from the plane passing through the O_1 STAT is parallel to the O axis, and the angle γ from the plane passing through the O axis parallel to the O_1 axis (these two planes obviously being parallel to each other)

and d is the distance between these planes or the distance between straight lines constituting a continuation of the O and O_1 axes. The sign in front of the d depends

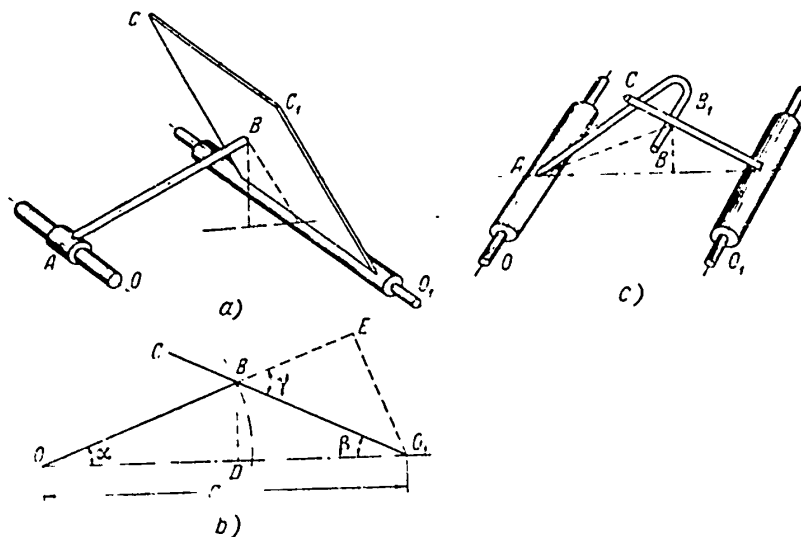


Fig. 76 - Rocker-Arm Connection in Which the Axes of the Driving Arm and the Rocker are Parallel

on the location of the O and O_1 axes.

Equation (9.22) gives

$$\frac{d\beta}{d\alpha} = \frac{\frac{a \cos \alpha}{c}}{1 + \frac{a^2 \sin^2 \alpha}{c^2}} = \frac{ac \cos \alpha}{c^2 - a^2 \sin^2 \alpha} \quad (9.24)$$

for the case in which the O and O_1 axes lie in a single plane. This is also obvious from the consideration that

$$\frac{d\beta}{d\alpha} = \frac{a}{c} \cos \alpha \cos^2 \beta,$$

while

$$\cos^2 \beta = \frac{c}{O_1 B}$$

STAT

and

$$O_1B = \sqrt{c^2 + a^2 \sin^2 \alpha}.$$

If $d \neq 0$ [eq.(9.23)], then

$$\frac{d\beta}{d\alpha} = \frac{\frac{a \cos \alpha}{c}}{1 + \left(\frac{a \sin \alpha \pm d}{c}\right)^2} = \frac{ac \cos \alpha}{c^2 + (a \sin \alpha \pm d)^2} \quad (9.25)$$

If the O and O_1 axes form any arbitrary angle δ , the following relation is obtained for the case in which O and O_1 lie in a single plane

$$\operatorname{tg} \beta = \frac{a \sin \alpha}{c - a \cos \delta \cos \alpha} \quad (9.26)$$

and for the case in which the O and O_1 axes do not lie in a single plane,

$$\operatorname{tg} \beta = \frac{a \sin \alpha \pm d}{c - a \cos \delta \cos \alpha} \quad (9.27)$$

In eqs.(9.26) and (9.27), c stands for the following value.

Let us drop a line from the point B , perpendicular to the O axis (the length of this perpendicular having the value a).

Let the base of this perpendicular be the point D (see Fig.77). The distance of this point from the axis O_1 is a value c_1 .

The projection of this distance c_1 onto a

plane passing through one of the axes (O or O_1) and parallel to the other (i.e., STAT one of the planes located at a distance of d), is equal to the value c . This quan-

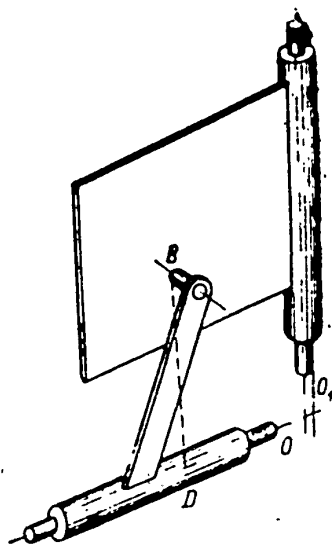


Fig.77 - Rocker-Arm Link in which the Axes of the Driving Lever and the Rocker are Normal to Each Other

tity is constant for every drive of this kind. For the case $\delta = \frac{\pi}{2}$, eqs.(9.26) and (9.27) yield eqs.(9.22) and (9.23). When $\delta = 0$, we obtain eq.(9.20) directly from eq.(9.26), while eq.(9.27) yields

$$\operatorname{tg} \beta = \frac{a \sin \alpha \pm d}{c - a \cos \alpha}. \quad (9.28)$$

It is not difficult to see that eq.(9.28) differs from eq.(9.20) only in that the angles β and α are counted in directions different from those in eqs.(9.20).

Actually, we may see from Fig.78 that

$$\operatorname{tg} \beta = \frac{BD}{O_1D} = \frac{a \sin \alpha + d}{c - a \cos \alpha}.$$

From eq.(9.26), we obtain

$$\begin{aligned} \frac{d\beta}{d\alpha} &= \frac{\frac{a \cos \alpha (c - a \cos \alpha \cos \delta) - a^2 \sin^2 \alpha \cos \delta}{(c - a \cos \alpha \cos \delta)^2}}{1 + \frac{a^2 \sin^2 \alpha}{(c - a \cos \alpha \cos \delta)^2}} = \\ &= \frac{a (c \cos \alpha - a \cos \delta)}{c^2 + a^2 - a^2 \cos^2 \alpha + a^2 \cos^2 \alpha \cos^2 \delta - 2ca \cos \alpha \cos \delta} = \\ &= \frac{a (c \cos \alpha - a \cos \delta)}{c^2 + a^2 - a^2 \sin^2 \delta \cos^2 \alpha - 2ca \cos \alpha \cos \delta}. \end{aligned} \quad (9.29)$$

And from eq.(9.27) we have

$$\begin{aligned} &\frac{\frac{a \cos \alpha (c - a \cos \alpha \cos \delta) - a \cos \delta \sin \alpha (a \sin \alpha \pm d)}{(c - a \cos \alpha \cos \delta)^2}}{1 - \left(\frac{a \sin \alpha \pm d}{c - a \cos \alpha \cos \delta} \right)^2} = \\ &= \frac{a (c \cos \alpha - a \cos \delta \mp d \sin \alpha \cos \delta)}{c^2 + a^2 + d^2 - a^2 \sin^2 \delta \cos^2 \alpha - 2ac \cos \alpha \cos \delta \pm 2ad \sin \alpha}. \end{aligned} \quad (9.30)$$

If $\delta = \frac{\pi}{2}$, then eq.(9.29) directly gives eq.(9.24), while eq. (9.30) gives eq.(9.25). If $\delta = 0$, eq.(9.29) directly yields eq.(9.21), and from eq.(9.30) we obtain

$$\frac{d\beta}{d\alpha} = \frac{a (c \cos \alpha - a \mp d \sin \alpha)}{c^2 + a^2 + d^2 - 2a (c \cos \alpha \mp d \sin \alpha)}.$$

It is not difficult to prove that we obtain the same expression if we determine this magnitude from eq.(9.28).

In eqs.(9.20) to (9.30), the angles β of rotation of the rocker and the quantity $\frac{d\beta}{d\alpha}$ are expressed as functions of the angle α of rotation of the lever OB.

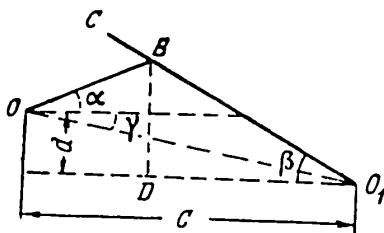


Fig.78 - See Derivation of Formula
for Dog Transmission with Parallel
Axes for Cases in which the Angles α
and β are not Counted along the
Straight Line OC_1

In the case in which the rocker is the driving element and the lever OB the driven element, it would be more convenient to express the angle α as a function of β and the angle $\frac{d\alpha}{d\beta}$ also as a function of the angle β .

However, we will not devote space to the derivation of these equations, since they not only are more complex but also far less widely applicable than those derived above. Designs in which the lever OB is the driver, and the rocker O_1C is

driven, are encountered much more frequently.

Chain Drives

Figures 79 and 80 illustrate chain drives.

A flexible chain consisting of very fine links (a sprocket chain) is attached at one end to the lever OB. The other end is wound around the pulley (1) which rotates on rotation of the lever OB. In Fig.79, the axis of the pulley is parallel to the axis of the lever OB, and in Fig.80 the axes O and O_1 are normal to each other.

Let us begin by examining the former variant of drive and assume that the chain is in a plane normal to the axes O and O_1 .

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Let us denote by l the distance BC of the free (developed) portion of the

chain, and by γ the angle between this portion of the chain and the plane passing through the axes O and O_1 . In that case, the angle β of rotation of the pulley (1)

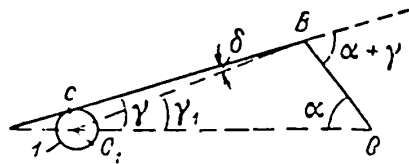


Fig. 79 - Diagram of Chain Drive in which the Axis of the Roller is Parallel to that of the Driving Lever

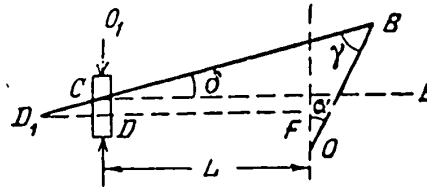


Fig. 80 - Diagram of Chain Drive with Axis of the Roller is Normal to that of the Driving Lever

will be equal to

$$\beta = \frac{l_0 - l}{r} + (\gamma - \gamma_0), \quad (9.31)$$

where l_0 and γ_0 are the magnitudes of l and γ when the drive is in its initial position;

r is the radius of the pulley (more exactly, the radius of the pulley plus half the thickness of the chain).

$\gamma = \gamma_1 - \delta$ (see Fig. 79), where γ_1 is determined from the equation

$$\operatorname{tg} \gamma_1 = \frac{a \sin \alpha}{c - a \cos \alpha}.$$

The quantity a denotes the length of the lever OB ; c is the distance OO_1 between the axes O and O_1 , while the angle δ is determined by the equality

$$\operatorname{tg} \delta = \frac{r}{l}.$$

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Thus,

$$\operatorname{tg} \gamma = \frac{la \sin \alpha - r(c - a \cos \alpha)}{l(c - a \cos \alpha) + ra \sin \alpha},$$

where

$$l = \sqrt{O_1 B^2 - r^2} = \sqrt{a^2 + c^2 - 2ac \cos \alpha - r^2}.$$

Substituting all these expressions in eq.(9.31), we obtain the value for the angle δ in the form of a rather complex function. It is simpler, therefore, to calculate the magnitude of l independently for each value of the angle α , than to do so for the values γ_1 and δ , and to write the resultant values into eq.(9.31).

The ratio $\frac{d\beta}{d\alpha}$ of the velocity of rotation of the pulley (1) to that of the lever OB can be established as soon as the ratio between the moments to be applied to the axes O and O_1 for equilibrating the system has been determined. Obviously,

$$\frac{M}{M_1} = \frac{a \sin(\alpha + \gamma)}{r},$$

where M is the moment relative to the axis O,

M_1 is the moment relative to the axis O_1 .

Thus,

$$\frac{d\beta}{d\alpha} = \frac{a}{r} \sin(\alpha + \gamma). \quad (9.32)$$

The quantity $\sin(\alpha + \gamma)$ can also be expressed as follows: It is clear from Fig.79 that

$$a \cdot \sin(\alpha + \gamma) = r + c \sin \gamma,$$

from which we derive

STAT

$$\frac{d\beta}{d\alpha} = 1 + \frac{c}{r} \sin \gamma \quad (9.33)$$

and

$$\sin \gamma = \sin (\gamma_1 - \delta).$$

After simple transformations, we obtain

$$\sin \gamma = \frac{a \sin \alpha \sqrt{c^2 + a^2 - 2ac \cos \alpha} - r^2 - r(c - a \cos \alpha)}{a^2 + c^2 - 2ac \cos \alpha}.$$

Substituting this expression in eq.(9.33), we obtain $\frac{d\beta}{d\alpha}$ as a function of α . Let us note, however, that, in calibrating it is easier to use eq.(9.32) where the $\frac{d\beta}{d\alpha}$ ratio is expressed as an implicit function of α .

In examining the drive illustrated in Fig.80, we will discuss only the simplest case, i.e., that in which the free portion of the chain BC is always normal to the O axis of rotation of the lever (i.e., the plane in which the B end of the lever CB undergoes displacement is not only parallel to the O_1 axis of the pulley, but also passes at a distance r from this axis, where r is the radius of the pulley plus half the thickness of the chain). In other words, in our case the point C is always in the same plane in which the B end of the lever CB undergoes displacement. But in this case, too, the precise relationship between the angle of rotation of the pulley (1) and the angle of rotation of the lever is quite complex.

If no friction existed between the chain and the surface of the pulley, the wound portion of the chain, whatever the position of the mechanism, would always lie on the pulley in a true helix with a constant angle of pitch δ , i.e., the angle between the free portion, BC of the chain and the straight line CE, normal to the O_1 axis. In this case, the problem would be greatly simplified, since the picture would be similar to that in a crank mechanism.

Actually, if the helical line is intentionally developed, the point D at which the end of the chain is attached to the pulley (1) coincides with the point D_1 , ^{STAT} lying along the straight line D_1F , parallel to the straight line CE (i.e., also nor-

mal to the axis O_1). Therefore, this assumption makes all the equations for crank drives, from (9.1) to (9.9) inclusive, applicable to the drive under discussion, except that in all these equations we must write $r\beta$ instead of x and $r \frac{d\beta}{d\alpha}$ instead of $\frac{dx}{d\alpha}$. Obviously, however, all these equations are applicable only to approximate calculation for this type of drive, since actually, friction prevents the wound portion of the chain from being arranged on the pulley (1) in the assumed manner for the most part, each basic segment will remain at the same angle to the generatrix of the pulley cylinder at which it happened to be at the moment of winding.

The result is that a rather complex curve is produced at the surface of the cylinder, so that the precise ratio between the rotation of the pulley (1) and of the lever OB becomes quite complex. For this reason, we will not derive this function here.

As far as the expression $\frac{d\beta}{d\alpha}$ as an implicit function of α is concerned, which, as indicated, may often serve for purposes of calibration, eq.(9.9) is completely and precisely applicable, i.e.,

$$\frac{d\beta}{d\alpha} \approx \frac{a \sin \gamma}{r \cos \delta},$$

where $\gamma = \frac{\pi}{2} - (\alpha + \delta)$ is the angle between the chain and the lever OB;

δ is the angle between the free (unwound) portion, BC of the chain and the line CE normal to the O_1 axis of the pulley (1). This equation is easily derived if we use the above concept to the effect that the velocities of rotation of the O_1 and O axes must be inversely proportional to the moments to be applied to these axes in order for the system to remain in equilibrium.

Gear Drives

Toothed drives are encountered in the transmitting mechanisms of instrument STAT chiefly as cylindrical gear wheels. Most frequently, toothed drives are used where it is necessary for some part (e.g., the pointer axis) to rotate through a large

angle.

The transmission ratio of a toothed drive with circular gear wheels is a constant and is equal to the ratio of the number of teeth in the gears or the ratio of the diameters of the addendum line (since these two ratios are identical when the gears mesh properly).

Section 2. Comparison of the Efficiencies of Articulated and Sliding Drives

In selecting the drive mechanism for obtaining a dial scale of the desired nature, one must not lose sight of the requirement that the type of selected transmitting mechanism must give rise to the least possible friction, reduced to the pickup of the instrument. This circumstance is quite significant where precision instruments are concerned.

We know that the transmitting mechanisms of instruments usually consist of a number of separate linkages. In selecting a drive mechanism, it should, if possible, be designed with linkages of the highest possible efficiency. A detailed comparison of the efficiencies of all the variants of transmitting linkages is quite complex, and we therefore do not derive these calculations here. In the present book, we mostly discuss general questions related to instrument design; however, it is useful to indicate that sliding linkages usually are lower in efficiency than hinged joints. Toward this end, we will compare the work lost to friction at least in the drives shown in Figs. 67 and 71, since it is entirely obvious that, the less the work lost to friction in the mechanism, the greater will be its efficiency.

Figure 67 depicts an ordinary crank mechanism, in which OA is a crank capable of rotating about the axis O, and BA is a connecting rod. The B end of the connecting rod is displaced along the line BD. Figure 71 presents a tangential sliding joint. Here the lever OB, rotating about the axis O, is always in contact with a point C, at which the part is displaced, this point C undergoing linear displacement in the direction CC_1 .

In comparing the work lost to friction in the two linkages under examination, we will consider the force F of the pressure exerted by the lever OB at the point C (in Fig.71) of the part it displaces, to be approximately equal to the force transmitted through the link AB in Fig.67.

Thus, in both cases the friction force is $F\mu$, where μ is the coefficient of friction. The radii of the pins by which the hinge linkages are effected will be considered identical at both hinges A and B in the hinge drive.

The angles δ , γ and α (see Fig.67) are correlated by the equation

$$\delta + \gamma = \frac{\pi}{2} - \alpha,$$

where γ is the angle between the lever OA and the link AB . Therefore, in a case in which the angles γ and δ either increase or decrease simultaneously, the sum of the simple angles of rotation in the hinges A and B (i.e., the sum of the absolute magnitudes $d\gamma$ and $d\delta$ of variation in the values of the angles γ and δ) is equal to the simple angle of rotation $d\alpha$ of the lever OA . This will be the case as long as γ is an acute angle, as is the case in Fig.67.

If γ is an obtuse angle, i.e., if a diminution in the angle γ results in an increase in the angle δ , then the angle $d\alpha$, through which the lever OA rotates, is equal to the difference between the angle of rotation in the hinge A and the angle of rotation in the hinge B . The total rotation in both hinges will be somewhat larger than the rotation of the lever $d\alpha$. However, as already indicated, in most drives, the variation in the angle δ is considerably less than the variation in the angles δ and α , and therefore the total rotation in the hinges A and B is not much greater than the rotation $d\alpha$ of the lever OA .

Thus, the number of paths passing through the hinges A and B at points in which friction is present, is $r d\alpha$ for the case of $\gamma < \frac{\pi}{2}$ and $kr d\alpha$ (where k is a magnitude very slightly greater than unity) for the case of $\gamma > \frac{\pi}{2}$. Therefore, the simple loss of work to friction du_1 will be

$$du_1 = F_{\mu} r d\alpha, \quad \text{at} \quad \gamma < \frac{\pi}{2}.$$

and

$$du_1 = k F_{\mu} r d\alpha, \quad \text{at} \quad \gamma > \frac{\pi}{2}.$$

The path passing through the lever OB (Fig. 71) at the point C, where the lever contacts the cam and the lever rotates through an angle $d\alpha$, is

$$d(OC) = \frac{d(OC)}{d\alpha} d\alpha.$$

However,

$$OC = \frac{a}{\cos \alpha},$$

and, therefore,

$$d(OC) = \frac{a \sin \alpha}{\cos^2 \alpha} d\alpha.$$

Thus, the simple work lost to friction du_2 in the given linkage will be

$$du_2 = F_{\mu} \frac{a \sin \alpha}{\cos^2 \alpha} d\alpha.$$

The ratio of work du_2 to the work du_1 is

$$\left. \begin{aligned} \frac{du_2}{du_1} &= \frac{a \sin \alpha}{r \cos^2 \alpha} \quad \text{where} \quad \gamma < \frac{\pi}{2}; \\ \frac{du_2}{du_1} &= \frac{a \sin \alpha}{kr \cos^2 \alpha} \quad \text{where} \quad \gamma > \frac{\pi}{2}. \end{aligned} \right\} \quad (9.34)$$

The radii of pins used in hinge-drive instruments is usually about 0.3 mm, ^{STAT} and the length a of the levers is rarely less than 6 mm (particularly in sliding joints). Therefore, the $\frac{a}{r}$ ratio may be taken as not less than 20. But it is not

hard to see that the expression $\frac{\sin \alpha}{\cos^2 \alpha}$ is already larger than $\frac{1}{20}$ in absolute magnitude when the angles α are relatively small.

For example, when $\sin \alpha = \frac{1}{8}$ (corresponding to $\alpha \approx 7^\circ 11'$), we have

$$\frac{\sin \alpha}{\cos^2 \alpha} = \frac{1.64}{8.63} = \frac{8}{63};$$

when $\sin \alpha = \frac{1}{4}$ (corresponding to $\alpha \approx 14^\circ 29'$)

$$\frac{\sin \alpha}{\cos^2 \alpha} = \frac{1.16}{4.15} = \frac{4}{15}$$

and when $\sin \alpha = \frac{1}{2}$ (corresponding to $\alpha = 30^\circ$)

$$\frac{\sin \alpha}{\cos^2 \alpha} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Thus, the simple work due to friction with a slide drive is not larger than the simple work due to friction in hinge drive only when the angle of α is less than two or three degrees, even when the lever a is small. In order for friction in a sliding linkage not to be larger, even on the average (over the entire operating range of the mechanism), than the friction of a hinge connection, it is essential that the variation in the magnitude of the angle α remains approximately within the limits of $+5^\circ$ to -5° , and that the lever a is quite small.

In other words, what is required is that the point C has a very limited travel: less travel than is usually the case in instruments. In addition, it would in this case always be necessary to adjust the transmission in such fashion that the angle α is zero at the mean working position of the mechanism. However, this would greatly limit the possibility of adjusting the mechanism.

However, sliding connections usually function at angles α that exceed the STAT its we have indicated, and therefore have a lower efficiency than hinge drives. In cases in which friction in the instrument mechanism is for some reason particularly

undesirable, an effort is always made to substitute hinged joints for sliding joints.

It is obvious that the dog transmission has an even lower efficiency than that of the instance illustrated in Fig. 71, since in a dog-driven linkage, the sliding of the contact points of the dogs occurs along both dogs; consequently, the work lost to friction will be the sum of the work so lost as a result of sliding along each of these dogs independently.

Note: Here it should be noted however, that in the operation of transmissions, the forces exerted on them create pressure not only in the hinges or the contact points between the parts of the mechanism, but also in the axes of rotation of the parts, or on the generatrices of the parts, if the motion of the parts is translational rather than rotational.

These latter pressures also give rise to loss of work due to friction in the axes of the parts. But these pressures do not differ greatly no matter whether the linkage is hinged or sliding. As a result, the total work lost to friction remains greater, with a sliding transmission, than the total work lost in hinge linkages. However, the ratio of the total work lost to friction in sliding linkages to the total work lost in hinge linkages will be somewhat less than indicated by eq. (9.34).

In addition, it should be noted that, in instrument mechanisms, the friction due to the weight of parts is usually not less but more than the friction due to the transmission of various forces through the mechanism (and it is only this friction, very often, that is taken into consideration in analyses of the efficiency of any mechanism). In actuality, the forces transferred by the transmitting mechanisms of instruments are usually very small.

Section 3. Brief Survey of Various Methods Facilitating the Selection of Transmit-STAT ting Mechanisms

Differences in the ratio of the rates of displacement of linked parts are pro-

vided in the majority of instrument transmitting linkages. In special cases, where it is required that the graduation of the instrument dial be proportional to the corresponding displacements of the pickup, this factor may complicate the design of the transmitting mechanism. But there are also opposite cases in which, by using the characteristic of transmitting linkages to which we have referred, a transmission mechanism is often designed with a more practical dial than the one obtained at a constant transmission ratio of the mechanism.

For example, it often happens that a displacement of the pickup, in accordance with a given change in the value being measured, is considerably greater in one portion of the instrument reading range than in another. If the transmission ratio of the mechanism were constant, a very nonuniform dial scale would result. In such cases, the above-mentioned property of transmission linkages can be used in designing a scale with significantly more uniform graduations. Sometimes it may even be required that an instrument have a scale with graduations of a predetermined ratio, i.e., that the instrument scale be uniform. This specification is usually met by the selection of an appropriate transmitting mechanism.

In all these cases we encounter the following: The displacement x of the portion of the pickup to which the transmission mechanism is connected, is some specific function of the value N to be measured, i.e., $x = f(N)$. However, it is required that the graduations of the instrument scale constitute some specific function of the magnitude to be measured N , perhaps significantly different from the function $f(N)$. In other words, the requirement is that $y = F(N)$, where y is the length of a segment of the scale from some initial reading to the graduation under examination.

Thus, if a uniform scale is required, then $y = kN$, where k is a constant determining the scale of the dial. If we have two equations, $x = f(N)$ and $y = F(N)$, then, eliminating the term N , it becomes possible to determine the required functional connection between the displacements x and y to be provided by the linkage,

i.e., we obtain an equation $y = \varphi(x)$ or $x = \varphi_1(y)$ which must satisfy the transmission mechanism of the instrument to the required degree.

The transmitting mechanism of an instrument consists of a number of linkages, whose most common variants are generally known. If for each of the linkages a functional connection exists between the displacements of the parts linked by the given linkage, it is possible to establish the functional relationship between the displacements x and y of the two extremes of the drive. This will give us an equation, $y = \theta(x)$.

The form of the function $\theta(x)$ depends both on the general design of the transmitting linkage and on the individual parameters of the linkages (sizes of parts, mutual arrangement of parts, initial angles between parts, etc.). Of course, in the majority of cases, the $\theta(x)$ function is not the same as the function $\varphi(x)$ and often differs greatly from the latter even in appearance. This is natural since, as we have seen from the examples, the $\theta(x)$ function is primarily a trigonometric function of some type. The $\varphi(x)$ function, however, may be of another type (logarithmic, algebraic, exponential, etc.).

Sometimes it is a very complicated matter accurately to reproduce a trigonometric function by means of linkage. In the majority of cases, a precise reproduction of the $\varphi(x)$ function is possible only by means of shaped cams. However, as already indicated, shaped cams are employed quite rarely in instrument mechanisms for the reason that it is virtually impossible to make two instruments absolutely identical in the processes of manufacture. This applies particularly to the pickups of the instruments: the characteristics of two springs (or pressure boxes) which would appear to have been produced by identical manufacture, are almost never completely identical. This usually results from the lack of uniformity of the metals from which the springs are made, as well as from unavoidable inaccuracies in their STAT manufacture.

The remaining parts of an instrument may also be not quite identical, and this

too, although to a lesser degree, results in unequal displacements of the mechanisms. Thus, if it were necessary, by means of a cam, to obtain precise identity of the functions $\theta(x)$ and $\varphi(x)$, the shapes of the cams in all instruments of the series would have to be different.

Note: In rare cases, shaped cams are used nonetheless to facilitate the design and calibration of the rest of the transmitting mechanism. In such cases, the cams are designed on the mean parameters of the pickup elements and other parts of the mechanism. Shaped cams are also encountered in unique instruments and special designs.

Thus, in most cases the equations for $\theta(x)$ and $\varphi(x)$ differ considerably in the form of their mathematical expression, but $\theta(x)$ is so selected that the numerical values of $\theta(x)$ and $\varphi(x)$ at values of x within the limits of operation of the instrument, differ very little (not more than by the magnitude indicated in the instrument specifications).

Most frequently, the designer starts by sketching the general diagram of the linkage, taking into consideration the general nature of the formulas for various transmission linkages calculated in various literature sources, as well as those derived on the basis of his own experience. Having established certain parameters of the mechanism (dimensions of parts, distances between parts, initial angles between parts, etc.), the designer checks at various points on the scale (numerically or graphically by the use of curves) whether the differences between the numerical values of the expressions $\theta(x)$ and $\varphi(x)$ differ to a sufficiently small degree. If the differences are excessive, the parameters of the mechanism have to be changed accordingly. In this case, it may sometimes be found that, no matter how much the parameters of the mechanism are changed, the required results will not be attained with the given type of drive. In such a case, it is necessary to change the overall design of the transmitting mechanism. STAT

Of course, when the parameters of the mechanism are derived in this manner,

every effort must be made to make certain that this derivation of the parameters does not impair the efficiency of the mechanism to any serious degree. Obviously, excessive rubbing is most undesirable in the transmitting mechanisms of an instrument. Yet this may readily occur if, in certain working positions of the mechanism, improper angles exist between the individual parts. In cases in which a particular type of dial (e.g., a uniform dial) is desirable but not absolutely essential, it is better to be satisfied with one that does not completely meet the original specifications (for example, one that is not completely uniform) as it is of course more important that the instrument function in good order.

There are several ways to select the type of transmission or certain of its parameters. These methods may either be general in nature and suitable for a variety of linkages or of special character and satisfactory only for a specific type of linkage. Certain of these methods are graphic. Graphic methods may, for example, facilitate the selection of the mechanism as a whole in cases in which the transmission mechanism consists of a number of consecutive simple linkages (cranks, the simplest types of cams, etc.), i.e., in a case in which

$$x_1 = \theta_1(x); \quad x_2 = \theta_2(x_1); \quad x_3 = \theta_3(x_2)$$

and

$$y = \theta_n(x_{n-1}).$$

where $x_1, x_2, \dots$ are the displacements of certain intermediate parts of the mechanism, and the above equations indicate the relationships between these displacements.

Sometimes the elimination of the intermediate values x_1, x_2 , etc. presents no great difficulties insofar as the derivation of the general expression $y = \theta(x)$ is concerned, but in particular instances this may be quite a problem. When this is the case, the widely used graphic method of investigating this combination of the relationships of the individual linkages of the mechanism is employed.

Thus, Fig.81 presents an example of this type of graphic presentation of the finite expression $y = \theta(x)$. Here a case is selected in which there is a series of three connections in the mechanism, i.e., in which we have, successively, a relationship between the displacements x , x_1 , x_2 , and y , where x_1 and x_2 are the displacements of two intermediate parts of the mechanism. Plotting separate graphs, I, II and III, representing the ratios $x_1 = \theta_1(x)$, $x_2 = \theta_2(x_1)$, and $y = \theta_3(x_2)$ for the various linkages comprising the mechanism, it is a very simple matter to find the curve for the ratio of the value of y to that of x .

In this connection, it is sensible to plot only the individual curves I, II

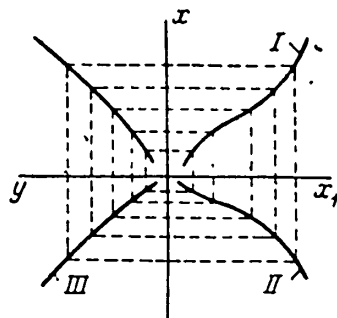


Fig.81 - Plot of Relationship Between Displacements of the First and Last Links in Mechanisms Consisting of Several Transmitting Linkages

and III comprising the graph and corresponding to the nonlinear relationship between the two displacements x_m and x_{m-1} of two adjacent parts, since in the case of a constant ratio between the rates of motion of any two adjacent parts of the mechanism (in a gear drive, for example), all that is needed is to select two series of graduations different in scale, along the corresponding axis of coordinates.

This method is also applicable in a case in which the mechanism contains only two linkages, with nonlinear transmission ratios. In a slightly more complex form, this method is also applicable to mechanisms with more than three linkages in which nonlinear transmission ratios are operative.

In certain methods of selecting transmitting mechanisms, the comparison of the magnitudes of the $\theta(x)$ and $\varphi(x)$ functions, i.e., of the specified and actually attained deflections of the pointer, is replaced by a comparison of the derivatives $\frac{d[\theta(x)]}{dx}$ and $\frac{d[\varphi(x)]}{dx}$ of these functions for various magnitudes in the operating

range of the instrument. In other words, a comparison is made of the transmission ratios of the mechanism desired and those actually obtained in accordance with specific values of x .

In these pages, we will not discuss all these methods in detail but will merely make certain practical considerations.

If the functional expression $y = \theta(x)$ for the first draft of the multiplying mechanism has already been derived and all that remains is to verify the degree to which the proposed tentative parameters of this drive satisfy the requirements thereof, it will often be more convenient to work not only with the expression $y = \theta(x)$, but also with $\frac{dy}{dx} = \frac{d[\theta(x)]}{dx}$, i.e., an expression for the total transmission ratio of the mechanism. This may prove convenient since the general appearance of the latter expression often facilitates an a priori solution of the question as to the direction in which some parameter of the mechanism has to be modified in order to obtain optimum functioning. Thus, the expression $\frac{dy}{dx}$ is recommended only as a supplementary one, while it is advisable to make the expression $y = \theta(x)$ the primary basis of choice.

It is true that some authors recommend the use of transmission ratio formulas as the basis of choice of a mechanism. However, the following reasons cast doubt on the advantages of this approach. In the majority of cases, the formulas expressing $\frac{dy}{dx}$ are not simpler than expressions for $y = \theta(x)$, but the opposite. Yet the $y = \theta(x)$ expression often gives a clearer idea of the suitability of a given linkage.

The point is that, when calculating the magnitudes of $\theta(x)$ and $\phi(x)$, corresponding to some magnitude of x , we immediately obtain a value for the discrepancy between these magnitudes; conversely, when we have only a relationship between the values of $\frac{dy}{dx}$ and x , we are compelled, in order to determine the degree to which the given linkage will deviate from the specifications for the nature of the instrument scale, to proceed to integrate the magnitude $\frac{dy}{dx} \cdot dx$ within the limits between the initial position of the mechanism and the position corresponding to the

magnitude of x .

Where a graphic relationship between the magnitudes $\frac{dy}{dx}$ and x is used, it is necessary to determine the area delimited by the curve for the given relationship, the abscissa, and the straight lines parallel to the ordinate, passing through the origin and the point x .

In complex mechanisms consisting of a number of separate linkages, and in cases in which it is difficult to obtain a general analytical expression for $y = \theta(x)$ or $\frac{dy}{dx}$, it would appear, at first glance, that the question is simplified by the fact that the general transmission ratio $\frac{dy}{dx}$ is equal to

$$\frac{dy}{dx} = \frac{dx_1}{dx} \frac{dx_2}{dx_1} \frac{dx_3}{dx_2} \dots \frac{dy}{dx_{n-1}}$$

In reality, however, this relief is more apparent than real, the reason being the following:

The quantities $\frac{dx_1}{dx}$, $\frac{dx_2}{dx_1}$, $\frac{dx_3}{dx_2}$... etc., for each position of the mechanism examined must all correspond to a single magnitude of x ; in other words, in order to obtain the value $\frac{d[\theta(x_m - 1)]}{dx_m - 1}$, corresponding to the position of the mechanism under examination, it is necessary to determine, first, the magnitude of $x_m - 1$ corresponding to the value of x , which means that it is again necessary to make use of an expression having the form $x_m - 1 = \theta_m - 1(x_m - 2)$. The end result is that, instead of facilitating the solution of the problem, it often is actually more complicated.

Section 4. Interpolation Method for Determining the Parameters of Transmitting Mechanisms

It is clear from the foregoing that, in a great many cases, the selection of transmitting mechanisms is not exactly a planned procedure, and often reduces to the following. Having set up some tentative diagram of the mechanism and arrived at preliminary parameters for its components, an effort is made to verify by calcula-

tion or graphically whether this will yield an adequate numerical agreement between the $\theta(x)$ and the $\varphi(x)$ functions, or their derivatives $\frac{d[\theta(x)]}{dx}$ and $\frac{d[\varphi(x)]}{dx}$ for various values within the operating range of the mechanism. Further, if there is inadequate agreement between these values, the parameters of the components of the mechanism are changed, or even the design itself, and the procedure of checking numerical agreement between the $\theta(x)$ and $\varphi(x)$ functions, or between the $\frac{d[\theta(x)]}{dx}$ and $\frac{d[\varphi(x)]}{dx}$ ratios, must again be performed for a number of points. In other words, the drive is obtained by a process of gradual numerical experimentation.

This method can hardly be regarded as convenient, and it would be more rational if some order were brought at least into the choice of parameters of transmitting linkages. With this purpose in mind, it is useful to set up in advance only a general design of the transmitting mechanism as a whole, and to determine the parameters in question directly from equations representing definite specifications for the mechanism, based on the condition that there must be the greatest possible numerical correspondence between the $\theta(x)$ and $\varphi(x)$ functions or their derivatives $\frac{d[\theta(x)]}{dx}$ and $\frac{d[\varphi(x)]}{dx}$.

Note: With regard to the preliminary tentative selection and subsequent changes in the basic design of the linkage, it is of course difficult to take equations of any kind as the point of departure, so that the only alternative is to employ the method of experimental selection indicated above.

It is obvious that equations such as we have been discussing may be written only for the number of parameters (or dimensions) that are subject to some variation in a given transmitting mechanism. Moreover, the method in question cannot be used successfully with all types of multiplying mechanisms, but in particular cases it is entirely applicable.

As the simplest kind of example of the application of this method, let us examine the selection of parameters for a transmitting mechanism consisting of a cam-type sinusoidal sliding joint (Figs. 69 and 70) and a toothed connection (sector and

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gear) used, for example, as the linkage of an ordinary centrifugal tachometer. In this mechanism, the value of the following three parameters is subject to change:

- a) the length of the lever OB, whose end rests in the plane of a cam moving linearly;
- b) the position of the axis of this lever relative to the initial position of the cam or, in other words, the original angle between the lever and the direction of motion of the cam, $\angle E_1$;
- c) the transmission ratio i between the sector and gear.

Accordingly, the linkage is subject to three specifications for the compilation of three equations. It is possible, for example, to specify the condition that the entire scale of the instrument from minimum reading N_0 to the maximum N_k , be contained within a specific angle ψ , in addition to which the following two specifications may be made: 1) If the given angle ψ is divided into three equal parts, the deflection of the pointer from its initial position (at an initial reading of N_0) by an angle of $\frac{\psi}{3}$ must correspond to some specific reading of the instrument N_1 , while the deflection of the pointer from its initial position by $\frac{2}{3}\psi$ must correspond to some other specific reading of N_2 (thus, it may be required that

$$N_1 - N_0 = N_2 - N_1 = N_k - N_2,$$

i.e., that the three equal changes in the instrument readings correspond to three equal arcs of the scale). Let us select the parameters of a linkage that satisfies these specifications.

Let us denote by α the angle between the lever OB and the line OD, normal to the direction EE_1 of the motion of the cam, and let us designate by l the length of the lever OB.

In the given instance, the three unknowns are the length l , the transmission ratio i of the sector to its gear, and the angle α_0 , corresponding to the initial reading of the instrument N_0 (this angle may be, and apparently is, usually negative). STAT

tive).

A calculation of the pickup element of the instrument yields the quantities h_0 , h_1 , h_2 , and h_k , which determine the positions of the cam corresponding to the measured values U_0 , U_1 , U_2 and U_k on displacement of the cam along a straight line EE_1 , calculating these distances from some initial point along this line. It then becomes possible to write the following equations:

$$\left. \begin{aligned} l \left[\sin \left(\alpha_0 + \frac{\psi}{3i} \right) - \sin \alpha_0 \right] &= h_1 - h_0; \\ l \left[\sin \left(\alpha_0 + \frac{2\psi}{3i} \right) - \sin \left(\alpha_0 + \frac{\psi}{3i} \right) \right] &= h_2 - h_1; \\ l \left[\sin \left(\alpha_0 + \frac{\psi}{i} \right) - \sin \left(\alpha_0 + \frac{2\psi}{3i} \right) \right] &= h_k - h_2. \end{aligned} \right\} \quad (9.35)$$

These three equations may be solved jointly for the unknowns l , α_0 and $\frac{\psi}{i}$, and several methods may be used. They may, for example, be solved as follows: Adding the three equations, we have

$$l \left[\sin \left(\alpha_0 + \frac{\psi}{i} \right) - \sin \alpha_0 \right] = h_k - h_0,$$

or

$$2l \cos \left(\alpha_0 + \frac{\psi}{2i} \right) \sin \frac{\psi}{2i} = h_k - h_0. \quad (9.36)$$

In exactly the same way, the second expression in eq.(9.35) may be presented in the form

$$2l \cos \left(\alpha_0 + \frac{\psi}{2i} \right) \sin \frac{\psi}{6i} = h_2 - h_1.$$

Dividing eq.(9.36) by this last equation, we obtain

$$\frac{\sin \frac{\psi}{2i}}{\sin \frac{\psi}{6i}} = \frac{h_k - h_0}{h_2 - h_1} = \frac{\sin \frac{\psi}{6i} \left(3 \cos^2 \frac{\psi}{6i} - \sin^2 \frac{\psi}{6i} \right)}{\sin \frac{\psi}{6i}} =$$

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$$= \frac{3}{2} \left(1 + \cos \frac{\psi}{3i} \right) - \frac{1}{2} \left(1 - \cos \frac{\psi}{3i} \right) = 1 + 2 \cos \frac{\psi}{3i},$$

from which we derive

$$\cos \frac{\psi}{3i} = \frac{1}{2} \left(\frac{h_k - h_0}{h_2 - h_1} - 1 \right). \quad (9.37)$$

If the magnitude i , obtained from the division of the value of angle ψ specified prior to the calculation, divided by the value of the angle $\frac{\psi}{i}$, proves to be an inconvenient fraction (i.e., one for which it is difficult to provide a rack and pinion with the required number of teeth), it is possible, in many instruments, to effect a slight change in the specification for ψ so as to obtain a handier transmission ratio i , especially since for most instruments the value of the angle ψ of the scale is actually specified only within rough limits (for example, it is usual to set the arc of the dial at about 300°). But the actual value of $\frac{\psi}{3i}$ must be permitted to remain the same for further calculation of the magnitudes of α_0 and l , as obtained from eq.(9.37).

Further, let us present the first and second equations in the system (9.35) in the form

$$2l \cos \left(\alpha_0 + \frac{\psi}{6i} \right) \sin \frac{\psi}{6i} = h_1 - h_0;$$

$$2l \cos \left(\alpha_0 + \frac{\psi}{2i} \right) \sin \frac{\psi}{6i} = h_2 - h_1.$$

From these two equations we obtain

$$\begin{aligned} (h_2 - h_1) \cos \alpha_0 \cos \frac{\psi}{6i} - (h_2 - h_1) \sin \alpha_0 \sin \frac{\psi}{6i} = \\ = (h_1 - h_0) \cos \alpha_0 \cos \frac{\psi}{2i} - (h_1 - h_0) \sin \alpha_0 \sin \frac{\psi}{2i}, \end{aligned}$$

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and from this it follows that

$$\operatorname{tg} \alpha_0 = \frac{(h_2 - h_1) \cos \frac{\psi}{6i} - (h_1 - h_0) \cos \frac{\psi}{2i}}{(h_2 - h_1) \sin \frac{\psi}{6i} - (h_1 - h_0) \sin \frac{\psi}{2i}} \quad (9.38)$$

Having determined $\frac{\psi}{3i}$ and α_0 , it becomes possible to determine the value of from any equation in the system (9.35).

Equation (9.37) shows that the solution for the unknown $\frac{\psi}{3i}$ is possible only if $\frac{h_k - h_0}{h_2 - h_1} < 3$. It should be noted that this requirement is satisfied in many cases.

Here it should be observed that the obtained solutions may sometimes be inapplicable since the magnitude of $\frac{h_k - h_0}{h_2 - h_1}$ in eq.(9.37) is exceedingly small so that the value of $\frac{\psi}{3i}$ is excessively large. It may then happen, in particular cases, that the angle $\alpha_{\max} = \alpha_0 + \frac{\psi}{i}$ is greater than $\frac{\pi}{2}$ or that the angle α_0 , being negative, is greater in absolute magnitude than $\frac{\pi}{2}$, or even that both angles $\alpha_{\max}$ and α_0 are greater in absolute magnitude than $\frac{\pi}{2}$. Obviously, in all these cases the function of linkage cannot be carried out.

But it may happen even more frequently that, while both angles α_0 and $\alpha_{\max}$ are smaller in absolute value than $\frac{\pi}{2}$, one of them or even both may not be sufficiently smaller than $\frac{\pi}{2}$. Furthermore, when the absolute values of the angle α are large, considerable friction may occur in the mechanism. It is generally desirable that the absolute magnitude of the angle α nowhere exceed approximately 40° .

If we emerge with one of the cases cited above, in which the solutions of eqs.(9.35) are unsatisfactory, this indicates that the objectives set for the transmission can hardly, if at all, be satisfied. In this case, the specifications must be changed somewhat, i.e., there must be some change in the ratios between magnitudes N_0 , N_1 , N_2 , and N_k (for example, a specific deviation from uniformity in the scale must be provided at the outset), and the equations in system (9.35) must then

be solved again with the new factors in the equation.

Here, however, it is necessary to make an observation. This arbitrary selection of points to be interpolated (i.e., the points M_0 , M_1 , M_2 , and M_k of the scale, in which the numerical values of the $\theta(x)$ and $\varphi(x)$ functions are precisely equal) cannot always satisfy the requirement that the values of these functions differ as little as possible at other instrument readings. Often, therefore, in the selection of parameters for transmitting mechanisms, after the numerical values of the $\varphi(x)$ function and the $\theta(x)$ function obtained from this interpolation have been checked at various points on the scale, the points of interpolation are shifted so as to obtain values of $\varphi(x)$ and $\theta(x)$ that are in closer agreement at all instrument readings. Then a new determination of linkage parameters is made by the indicated method of interpolation, and the procedure is repeated. (When this approach is followed, one may move not only the intermediate points of interpolation M_1 , M_2 , etc., but also the extreme interpolation points M_0 and M_k , i.e., these extreme points need not coincide with the extreme readings of the instrument).

Recently, certain methods were developed to replace the guesswork in this shifting of the interpolation points by certainty so that, on each occasion, the values of $\varphi(x)$ and $\theta(x)$ can be brought into closer agreement (see, for example, Ye.P. Novodvorskii and I.Sh.Pinsker, Bibl.18, and others).

In addition, even in the first interpolation (which is often confined within narrow bounds), it is best to select the interpolation points not entirely by guesswork; the more so as methods exist for a preliminary selection of the interpolation points at which the choice, in very many cases, yields results close to the optimum. Thus, for example, Chebyshev proposed the following:

Taking a line segment depicting, for a given scale, the range of instrument readings from its minimum $M_{\min}$ to its maximum $M_{\max}$, a circle is described around this segment as diameter (Fig.82), and a rectilinear polygon with $2n$ sides is inscribed therein, n being the number of interpolation points (for example, in the in-

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stance analyzed above there were four such points: N_0, N_1, N_2 , and N_k); here the method of inscription is such that two sides of this polygon are normal to the diameter $N_{min} N_{max}$. The bases N_0, N_1, N_2 etc., of the perpendiculars dropped from the apexes of angles A_0, A_1, A_2 etc., to the diameter $N_{min} N_{max}$, are used, in Chebyshev's method, as interpolation points in determining the parameters of the mechanism. Experience has shown this method of choosing points of interpolation to be satisfactory in a very large number of cases.

Note: If the extreme interpolation points, N_0 and N_k do not correspond to the extreme points N_{min} and N_{max} on the instrument scale (in the above instance, we made use of the special case in which these points agree) then, knowing the total angle ψ_c covered by the scale from N_{min} to N_{max} and the ratio of the range from N_{min} to N_{max} to that from N_0 to N_k , it is easy to calculate the angle ψ of the arc of the scale corresponding to the range from N_0 to N_k , required for proper selection of the parameters of the mechanism.

When the interpolation points are chosen by this method, which is not entirely

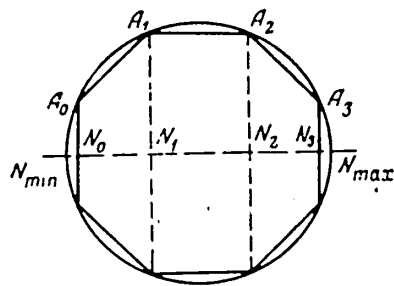


Fig.82 - Chebyshev's Method for Approximated Determination of Interpolation Points

arbitrary, and particularly when these points are then shifted to obtain still closer agreement between the values of the functions $\phi(x)$ and $\theta(x)$, it will very often happen that the ratio between the arcs of the dial corresponding to the segments between the interpolation points, will be much less convenient than was the case in the instance under examination. Often this somewhat complicates the combined solution of the resultant equations. STAT

Thus, if, in the example examined above, the angle of the scale corresponding to the range $N_0 - N_1$ is not equal to $\frac{\psi}{3}$ but to some quantity $n_1\psi$, and the angle cor-

responding to the range $\psi_0 - \psi_2$ is not equal to $\frac{2}{3}\psi$, but to some magnitude $n_2\psi$ in which n_1 and n_2 are constants of any value (less than unity), the system of equations

$$\left. \begin{aligned} l \left[\sin \left(\alpha_0 + n_1 \frac{\psi}{i} \right) - \sin \alpha_0 \right] &= h_1 - h_0; \\ l \left[\sin \left(\alpha_0 + n_2 \frac{\psi}{i} \right) - \sin \alpha_0 \right] &= h_2 - h_0; \\ l \left[\sin \left(\alpha_0 + \frac{\psi}{i} \right) - \sin \alpha_0 \right] &= h_k - h_0 \end{aligned} \right\} \quad (9.39)$$

is no longer as convenient for combination solution as the systems in eq.(9.35). It is obvious that the clumsy form of these equations is due to the fact that one of the unknowns to be determined is the quantity i (or the angle $\frac{\psi}{i}$). It should be noted that the latter applies in the majority of cases also to other forms of transmitting linkages. In almost every case, if the total angle $\frac{\psi}{i}$ of operation of the linkage in the range from ψ_0 to ψ_k is not established at the beginning, the problem will be greatly complicated. If we use as the wanted unknowns any other parameters (but not, of course, in the example under examination, since nothing further can be determined in this linkage) we complicate matters much less.

A combined solution of the equations in system (9.39) may be accomplished as follows: From the first and third equations we have

$$\begin{aligned} (h_k - h_0) \left(\sin \alpha_0 \cos n_1 \frac{\psi}{i} + \cos \alpha_0 \sin n_1 \frac{\psi}{i} - \sin \alpha_0 \right) &= \\ = (h_1 - h_0) \left(\sin \alpha_0 \cos \frac{\psi}{i} + \cos \alpha_0 \sin \frac{\psi}{i} - \sin \alpha_0 \right), \end{aligned}$$

from which we obtain

$$\operatorname{tg} \alpha_0 = \frac{(h_k - h_0) \sin n_1 \frac{\psi}{i} - (h_1 - h_0) \sin \frac{\psi}{i}}{(h_k - h_0) \left(1 - \cos n_1 \frac{\psi}{i} \right) - (h_1 - h_0) \left(1 - \cos \frac{\psi}{i} \right)} \quad (9.40) \quad \text{STAT}$$

Analogously, the second and third equations yield

$$\operatorname{tg} \alpha_0 = \frac{(h_k - h_0) \sin n_2 \frac{\psi}{i} - (h_2 - h_0) \sin \frac{\psi}{i}}{(h_k - h_0) \left(1 - \cos n_2 \frac{\psi}{i}\right) - (h_2 - h_0) \left(1 - \cos \frac{\psi}{i}\right)} \quad (9.40a)$$

Equating the two above expressions for $\tan \alpha_0$, we obtain an equation for one unknown. However, this will be quite cumbersome and will not permit an analytical solution in the general case. However, if necessary it may be solved graphically in the usual way, by plotting, along the points, a curve for a function representing the difference between the right sides of eqs. (9.40) and (9.40a) versus the angle $\frac{\psi}{i}$ (the magnitude of $\frac{\psi}{i}$ being plotted on the abscissa and the corresponding values of the functions on the ordinate). The intersection of the curve with the abscissa, as we know, is the solution of the equation.

As another example, we may examine the determination of the parameters of a mechanism consisting of a crank drive and a rack-and-pinion drive, a type of mechanism that is frequently encountered in instruments. Here (see Fig. 67) the parameters of the linkage subject to variation, are as follows

- a) the length a of the crank OA ;
- b) the magnitude d of the offset OO_1 ;
- c) the magnitude c_0 of the distance O_1B_0 , where B_0 is the position of the point B at the initial position of the mechanism;
- d) the length b of the link AB , or the magnitude α_0 of the angle α when the mechanism is in its original position (α being the angle between the line OD normal to the direction BD of motion of the slide B and the lever OA). It is obvious that, at the dimensions of the enumerated parameters OA , B_0O_1 , and OC_1 , the magnitudes of b and α_0 are functionally related in some definite fashion, so that, assigning a value to one of them, we obtain a definite value for the other;

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- e) the total value of the range of variation in the angle α (or in other words the transmission ratio i of the sector to the gear, as assigned to the total angle ψ).

If we assign at the outset the total variation range of the angle α (as indicated, a determination of this value as an unknown from the general linkage equations usually complicates the problem greatly), we are left with four parameters that can be determined by interpolation, so that we can assign five interpolation points.

In the given case we may write (see Fig.67) the following five equations, introducing temporarily into this system of equations, in addition to α_0 , a fifth unknown (not independent), i.e., the length b of the link AB in

$$\left. \begin{aligned} [a \sin \alpha_0 + c_0]^2 + [a \cos \alpha_0 - d]^2 &= b^2; \\ [a \sin (\alpha_0 + \Delta \alpha_1) + c_0 - x_1]^2 + [a \cos (\alpha_0 + \Delta \alpha_1) - d]^2 &= b^2; \\ [a \sin (\alpha_0 + \Delta \alpha_2) + c_0 - x_2]^2 + [a \cos (\alpha_0 + \Delta \alpha_2) - d]^2 &= b^2; \\ [a \sin (\alpha_0 + \Delta \alpha_3) + c_0 - x_3]^2 + [a \cos (\alpha_0 + \Delta \alpha_3) - d]^2 &= b^2; \\ [a \sin (\alpha_0 + \Delta \alpha_4) + c_0 - x_4]^2 + [a \cos (\alpha_0 + \Delta \alpha_4) - d]^2 &= b^2, \end{aligned} \right\} \quad (9.41)$$

in which $\Delta \alpha_1, \Delta \alpha_2, \Delta \alpha_3$, and $\Delta \alpha_4$ are the angles of rotation of the crank from its initial position, corresponding to the first interpolation point (at the starting position), to positions corresponding to the second, third, fourth, and fifth points of interpolation;

x_1, x_2, x_3 , and x_4 are, correspondingly, the displacements of the slide in the direction BQ_1 .

Denoting the value $a \sin \alpha_0$ by m and $a \cos \alpha_0$ by n , we have

$$\left. \begin{aligned} (m + c_0)^2 + (n - d)^2 &= b^2; \\ (m \cos \Delta \alpha_1 + n \sin \Delta \alpha_1 + c_0 - x_1)^2 + (n \cos \Delta \alpha_1 - m \sin \Delta \alpha_1 - d)^2 &= b^2; \\ (m \cos \Delta \alpha_2 + n \sin \Delta \alpha_2 + c_0 - x_2)^2 + (n \cos \Delta \alpha_2 - m \sin \Delta \alpha_2 - d)^2 &= b^2; \\ (m \cos \Delta \alpha_3 + n \sin \Delta \alpha_3 + c_0 - x_3)^2 + (n \cos \Delta \alpha_3 - m \sin \Delta \alpha_3 - d)^2 &= b^2; \\ (m \cos \Delta \alpha_4 + n \sin \Delta \alpha_4 + c_0 - x_4)^2 + (n \cos \Delta \alpha_4 - m \sin \Delta \alpha_4 - d)^2 &= b^2. \end{aligned} \right\} \quad (9.42)$$

Thus we have five equations with five unknowns m, n, c_0, d , and b . Squaring the polynomials, as indicated in the equations, and then subtracting the first equation from the four others, we obtain the following four equations with four unknowns:

$$\begin{aligned}
 & 2(1 - \cos \Delta \alpha_1) mc_0 - 2 \sin \Delta \alpha_1 nc_0 - 2(1 - \cos \Delta \alpha_1) nd - \\
 & - 2 \sin \Delta \alpha_1 md + 2x_1 \cos \Delta \alpha_1 m + 2x_1 \sin \Delta \alpha_1 n + 2x_1 c_0 - x_1^2 = 0; \\
 & 2(1 - \cos \Delta \alpha_2) mc_0 - 2 \sin \Delta \alpha_2 nc_0 - 2(1 - \cos \Delta \alpha_2) nd - \\
 & - 2 \sin \Delta \alpha_2 md + 2x_2 \cos \Delta \alpha_2 m + 2x_2 \sin \Delta \alpha_2 n + 2x_2 c_0 - x_2^2 = 0; \\
 & 2(1 - \cos \Delta \alpha_3) mc_0 - 2 \sin \Delta \alpha_3 nc_0 - 2(1 - \cos \Delta \alpha_3) nd - \\
 & - 2 \sin \Delta \alpha_3 md + 2x_3 \cos \Delta \alpha_3 m + 2x_3 \sin \Delta \alpha_3 n + 2x_3 c_0 - x_3^2 = 0; \\
 & 2(1 - \cos \Delta \alpha_4) mc_0 - 2 \sin \Delta \alpha_4 nc_0 - 2(1 - \cos \Delta \alpha_4) nd - \\
 & - 2 \sin \Delta \alpha_4 md + 2x_4 \cos \Delta \alpha_4 m + 2x_4 \sin \Delta \alpha_4 n + 2x_4 c_0 - x_4^2 = 0.
 \end{aligned} \tag{9.43}$$

In each of these four equations, the coefficients of the product of the unknowns mc_0 and nd are equal. Likewise, the coefficients of the products of the unknowns nc_0 and md are equal to each other. Therefore, when the first terms (containing the product mc_0) are eliminated from the equations, this automatically excludes the third terms (containing the product nd). The same may be stated with regard to the second and fourth terms in the equations (containing the products nc_0 and md).

Since it is possible to eliminate three terms from the four equations, eliminating the fifth term (containing only the unknown m) in addition to the first and second, will yield an equation of the following form

$$\begin{aligned}
 & \begin{vmatrix} 1 - \cos \Delta \alpha_1 & \sin \Delta \alpha_1 & x_1 \cos \Delta \alpha_1 & x_1 \sin \Delta \alpha_1 \\ 1 - \cos \Delta \alpha_2 & \sin \Delta \alpha_2 & x_2 \cos \Delta \alpha_2 & x_2 \sin \Delta \alpha_2 \\ 1 - \cos \Delta \alpha_3 & \sin \Delta \alpha_3 & x_3 \cos \Delta \alpha_3 & x_3 \sin \Delta \alpha_3 \\ 1 - \cos \Delta \alpha_4 & \sin \Delta \alpha_4 & x_4 \cos \Delta \alpha_4 & x_4 \sin \Delta \alpha_4 \end{vmatrix} n + \\
 & + \begin{vmatrix} 1 - \cos \Delta \alpha_1 & \sin \Delta \alpha_1 & x_1 \cos \Delta \alpha_1 & x_1 \\ 1 - \cos \Delta \alpha_2 & \sin \Delta \alpha_2 & x_2 \cos \Delta \alpha_2 & x_2 \\ 1 - \cos \Delta \alpha_3 & \sin \Delta \alpha_3 & x_3 \cos \Delta \alpha_3 & x_3 \\ 1 - \cos \Delta \alpha_4 & \sin \Delta \alpha_4 & x_4 \cos \Delta \alpha_4 & x_4 \end{vmatrix} c_0 - \\
 & - \frac{1}{2} \begin{vmatrix} 1 - \cos \Delta \alpha_1 & \sin \Delta \alpha_1 & x_1 \cos \Delta \alpha_1 & x_1^2 \\ 1 - \cos \Delta \alpha_2 & \sin \Delta \alpha_2 & x_2 \cos \Delta \alpha_2 & x_2^2 \\ 1 - \cos \Delta \alpha_3 & \sin \Delta \alpha_3 & x_3 \cos \Delta \alpha_3 & x_3^2 \\ 1 - \cos \Delta \alpha_4 & \sin \Delta \alpha_4 & x_4 \cos \Delta \alpha_4 & x_4^2 \end{vmatrix} = 0. \tag{9.44} \text{ STAT}
 \end{aligned}$$

However, if the first, second, and sixth terms are eliminated, we derive an equation of the form

$$\begin{aligned}
 & \begin{vmatrix} 1 - \cos \Delta \alpha_1 & \sin \Delta \alpha_1 & x_1 \cos \Delta \alpha_1 & x_1 \sin \Delta \alpha_1 \\ 1 - \cos \Delta \alpha_2 & \sin \Delta \alpha_2 & x_2 \cos \Delta \alpha_2 & x_2 \sin \Delta \alpha_2 \\ 1 - \cos \Delta \alpha_3 & \sin \Delta \alpha_3 & x_3 \cos \Delta \alpha_3 & x_3 \sin \Delta \alpha_3 \\ 1 - \cos \Delta \alpha_4 & \sin \Delta \alpha_4 & x_4 \cos \Delta \alpha_4 & x_4 \sin \Delta \alpha_4 \end{vmatrix} m + \\
 & + \begin{vmatrix} 1 - \cos \Delta \alpha_1 & \sin \Delta \alpha_1 & x_1 & x_1 \sin \Delta \alpha_1 \\ 1 - \cos \Delta \alpha_2 & \sin \Delta \alpha_2 & x_2 & x_2 \sin \Delta \alpha_2 \\ 1 - \cos \Delta \alpha_3 & \sin \Delta \alpha_3 & x_3 & x_3 \sin \Delta \alpha_3 \\ 1 - \cos \Delta \alpha_4 & \sin \Delta \alpha_4 & x_4 & x_4 \sin \Delta \alpha_4 \end{vmatrix} c_0 - \\
 & - \frac{1}{2} \begin{vmatrix} 1 - \cos \Delta \alpha_1 & \sin \Delta \alpha_1 & x_1^2 & x_1 \sin \Delta \alpha_1 \\ 1 - \cos \Delta \alpha_2 & \sin \Delta \alpha_2 & x_2^2 & x_2 \sin \Delta \alpha_2 \\ 1 - \cos \Delta \alpha_3 & \sin \Delta \alpha_3 & x_3^2 & x_3 \sin \Delta \alpha_3 \\ 1 - \cos \Delta \alpha_4 & \sin \Delta \alpha_4 & x_4^2 & x_4 \sin \Delta \alpha_4 \end{vmatrix} = 0 \quad (9.45)
 \end{aligned}$$

Thus, we have two equations of the form

$$D_1 n + D_2 c_0 - \frac{1}{2} D_3 = 0;$$

$$D_1 m + D_4 c_0 - \frac{1}{2} D_5 = 0,$$

where the symbols D_1 , D_2 , D_3 , D_4 , and D_5 represent the determinants in eqs. (9.44) and (9.45).

From this we obtain

$$\left. \begin{aligned} n &= -\frac{D_2}{D_1} c_0 + \frac{1}{2} \frac{D_3}{D_1}; \\ m &= -\frac{D_4}{D_1} c_0 + \frac{1}{2} \frac{D_5}{D_1}. \end{aligned} \right\} \quad (9.46)$$

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Substituting these equations in any two equations of the system (9.43), we obtain two equations of the second power, in the form

$$\left. \begin{aligned} A_1 c_0 d + B_1 c_0^2 + k_1 c_0 + h_1 d + L_1 &= 0; \\ A_2 c_0 d + B_2 c_0^2 + k_2 c_0 + h_2 d + L_2 &= 0, \end{aligned} \right\} \quad (9.47)$$

where the values $A_1, A_2, B_1, B_2, k_1, k_2, h_1, h_2, L_1$ and L_2 are specific constant factors.

A combined solution of these equations obviously results in the solution of an equation of the third power, from which the unknown c_0 is found. Each of three solutions of this equation corresponds to one solution for d , as is evident from the equations in the system (9.47), one solution for n and one for m , as is evident from eqs. (9.44) and (9.45). Thus, we obtain three systems of solutions for eqs. (9.43).

In a great many mechanisms, the length of the link b , for design reasons, is intentionally made considerably longer than the crank a . In addition, for a number of reasons the most important of which is prevention of an increase in the errors due to friction in the instrument, very acute or obtuse angles between crank and link are usually not allowed in these mechanisms.

However, given a high ratio of the length of the link to the crank (which is usually the case in our instruments), certain variations in the length $B_0 C_1$ have comparatively little effect on the nature of the gradual variation in the transmission ratio of a mechanism in operation. Therefore, if this length is considered one of the wanted unknowns in the system of equations we are to compile, and if, in view of this, an additional requirement is made on the linkage (to obtain a new equation), we are running the risk of obtaining a system whose solutions are not suited to the given design (i.e., one with a very short link, or very acute or obtuse angles between link and crank), or even one whose solutions are completely inapplicable (i.e., imaginary or negative parameters where positive results are obviously required, STAT sines or cosines greater than unity, etc.). This being the case, it is preferable in the case in question to assign the magnitude of $B_0 C_1$ at the outset, simply on the

basis of considerations of size.

A similar statement can be made relative to the magnitude d of the offset CC_1 , whose variation also has little effect on the nature of the change in the transmission mechanism if the link AB is to have a sufficient length. Therefore, here too, it is not rational to make further demands on the transmission to determine the magnitude of CC_1 , and it is best to assign it at the outset. This being the case, of course, size considerations may be of somewhat less assistance than distance determinations B_0C_1 ; in addition to these considerations, it is desirable to make a preliminary analysis of the length of the crank OA .

Thus, if the dimensions of B_0C_1 and CC_1 are assigned, what remains is to determine three independent parameters if the total range of variation in the angle α is included among them.

Here, as in the above example, we may specify that three segments of the dial (in the range of readings from ψ_0 to ψ_1 , ψ_1 to ψ_2 , and ψ_2 to ψ_3) should be at a specific ratio. This constitutes two requirements for the linkage. The third specification is the usual one, i.e., that the total angle of rotation of the gear be equal to a specific value (the dial angle).

If, for an approximated solution of the problem, we ignore the variation in length of the projection of the link onto the direction of motion of the slide, since this change in the length of the projection of the link has little effect at a sufficiently large ratio of the length of the link to that of the crank, then, for this linkage as well, we can make use of the above method of determining the parameters of a sinusoidal sliding joint. However, a more precise determination is possible by allowing for variations in the length of the projection of the rod.

In the following, we will give a method for the combined solution of a system of equations, based on the three requirements indicated above. However, as shown below, in this situation, after elimination of the two unknowns, a rather complex equation is obtained, which can be solved graphically when necessary. In view of

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this, we will first solve a simpler problem, which may be satisfactory in many cases for selecting the linkage of an instrument, to wit: Let us assign in advance the transmission ratio, i , of the rack to the pinion. By so doing, we assign a specific quantity $\frac{\psi}{i} = \Delta\alpha_c$ for the total angle of rotation of the sector.

In this case only two unknowns remain. Therefore, we can derive only two equations, based on the requirement that two segments of the dial be equal to two specific angular magnitudes. In other words, it is necessary to divide the entire scale (and therefore the total angle $\Delta\alpha_c$ of rotation of the crank) into two specific parts (e.g., into two equal halves) and to specify that the first portion of the dial correspond to the readings of the instrument from N_0 to some particular reading N_1 , and the second portion of the dial correspond to the readings from N_1 to the maximum reading of the instrument N_k .

In this connection, let us note the following: This simplified method of determining a linkage, by dividing the range of the instrument readings into two portions rather than three, may in various cases be of little value in designing a dial of the type desired (such as a dial with equidistant graduations). This usually occurs when the relationship between the magnitude being measured and the travel of the pickup element is such that, in order for the dial to be of the type desired, the mechanism must provide a gradual reduction at the start, followed by a gradual multiplication (or, conversely, a mechanism that amplifies first and then reduces).

It is quite obvious that, in this case, satisfaction of the requirement that both parts of the dial are at a specific ratio to each other does not in itself provide a solution, and it is essential that all three scales have a specific mutual relationship. This statement is fully applicable, for example, to the centrifugal tachometer, if we attempt, by means of our choice of transmitting mechanisms, to produce the most uniform possible scale. In these tachometers, the ratio $\frac{dx}{dN}$, STAT (where dx is the simple displacement of a movable coupling corresponding to the simple change dN in the number of revolutions being measured) is comparatively small at

first, when the instrument readings are low. Subsequently it increases, and at high readings again declines (the dial usually having smaller graduations at the beginning and the end of the scale). In this situation, equality of the two halves of the dial, corresponding to two identical reading ranges, is of no value to us.

However, in a case in which it is required that the amplifying ratio increase or decrease over the entire range of instrument readings, i.e., when the curve of the required ratio of the travel of the arrow to the travel of the pickup is to show either a continuous concavity or a continuous convexity in a given direction (which is the case in many aircraft instruments, such as air temperature indicators, air-speed indicators, altimeters, pressure-and-suction gages, etc.), we may guide ourselves by the following considerations.

In virtually all the linkages generally used in our instruments (including those used in cranks), the curves showing the relationship between the motion of the last and first links in the mechanism have only one point of inflection. This point is approximately at the middle of the curve (when the transmitting linkage is in the center position), and the curve has comparatively little curvature (it has a large radius of curvature) for a considerable distance adjacent to the point of inflection. In addition, in mechanisms consisting of this type of joint and a rack-and-pinion drive, it is almost always essential to specify a relatively small angle of total rotation of the axis of the sector (in the given case, the axis is that of the crank), to reach a satisfactory transmission ratio in the drive.

Therefore, if it is necessary for the transmission ratio of the mechanism to show either a constant increase or a constant decrease, and if the above specification is extended to read that only two parts of the dial reveal a specific relationship, then, as a practically uniform rule, it will follow that the transmission functions along a segment of the curve lying completely or almost entirely on one side^{STAT} of the inflection point of the curve for the transmission ratio, i.e., use is made either of the portion of the transmission curve revealing either concavity or con-

vexity alone on one side or the other.

Under these conditions, it may happen even more frequently that when there is coincidence of only three points on two curves (i.e., on the curves of the required and actual relationships of the linkages), these curves will diverge at other points to a degree permissible for the given case. Therefore, the simplified solution of the problem under study, in which the value of i is assigned in advance, may often yield satisfactory results.

In such cases we may again make use of expressions analogous to those in eq. (9.43), but there will be only two, of course; since there are only two unknowns m and n and only two segments of the scale. Moreover, these are equations of the first power, since the values of c_0 and d are given.

Such a system of two linear equations is readily solved in terms of the unknowns m and n , and α_0 may be determined therefrom. In addition, if we exclude from these two equations the constants $2x_1c_0 - x_1^2$ and $2x_2c_0 - x_2^2$, we obtain

$$\begin{aligned} & (2c_0x_2 - x_2^2) \{ [c_0(1 - \cos \Delta\alpha_1) - d \sin \Delta\alpha_1 + x_1 \cos \Delta\alpha_1] m - \\ & \quad - [d(1 - \cos \Delta\alpha_1) + c_0 \sin \Delta\alpha_1 - x_1 \sin \Delta\alpha_1] n \} = \\ & = (2c_0x_1 - x_1^2) \{ [c_0(1 - \cos \Delta\alpha_2) - d \sin \Delta\alpha_2 + x_2 \cos \Delta\alpha_2] m - \\ & \quad - [d(1 - \cos \Delta\alpha_2) + c_0 \sin \Delta\alpha_2 - x_2 \sin \Delta\alpha_2] n \}, \end{aligned}$$

from which, substituting $c_0 - x_1 = c_1$ and $c_0 - x_2 = c_2$ in passing (with the result that $2c_0x_1 - x_1^2 = c_0^2 - c_1^2$ and $2x_2c_0 - x_2^2 = c_0^2 - c_2^2$), we obtain

$$\begin{aligned} \operatorname{tg} \alpha_0 &= \frac{m}{n} = \\ &= \frac{(c_0^2 - c_2^2)(d - d \cos \Delta\alpha_1 + c_1 \sin \Delta\alpha_1) - (c_0^2 - c_1^2)(d - d \cos \Delta\alpha_2 + c_2 \sin \Delta\alpha_2)}{(c_0^2 - c_2^2)(c_0 - c_1 \cos \Delta\alpha_1 - d \sin \Delta\alpha_1) - (c_0^2 - c_1^2)(c_0 - c_2 \cos \Delta\alpha_2 - d \sin \Delta\alpha_2)}. \quad (9.48) \end{aligned}$$

If, for a more precise solution of the problem, we also determine the total magnitude $\Delta\alpha_c$ of rotation of the crank between its two extreme positions, result: STAT from two extreme interpolation points, without having assigned this value in advance (or, in other words, if we also determine the transmission ratio between the rack

and the pinion of the instrument), then obviously we may write the four expressions of eq.(9.42) (since we are now comparing three segments of the dial) in which $\Delta\alpha_1 = n_1\Delta\alpha_c$; $\Delta\alpha_2 = n_2\Delta\alpha_c$, and $\Delta\alpha_3 = \Delta\alpha_c$, where n_1 and n_2 denote specific given values and c_0 and d are also given values, so that the wanted unknowns will be m , n , $\Delta\alpha_c$, and b .

Therefore, we now obtain three equations of the type of eq.(9.43) (the only difference being that $\Delta\alpha_1$ is replaced by the expression $n_1\Delta\alpha_c$, and $\Delta\alpha_2$ by $n_2\Delta\alpha_c$, and finally $\Delta\alpha_3$ by $\Delta\alpha_c$). Determining $\tan \alpha_0$ in the above manner, from the first and third equations, we obtain.

$$\begin{aligned} \operatorname{tg} \alpha_0 &= \\ &= \frac{(c_0^2 - c_3^2) [d - d \cos (n_1 \Delta\alpha_c) + c_1 \sin (n_1 \Delta\alpha_c)] - (c_0^2 - c_1^2) [d - d \cos \Delta\alpha_c + c_3 \sin \Delta\alpha_c]}{(c_0^2 - c_3^2) [c_0 - c_1 \cos (n_1 \Delta\alpha_c) - d \sin (n_1 \Delta\alpha_c)] - (c_0^2 - c_1^2) [c_0 - c_3 \cos \Delta\alpha_c - d \sin \Delta\alpha_c]} \end{aligned}$$

However, if we determine $\tan \alpha_0$ from the second and third equations; we will obtain

$$\begin{aligned} \operatorname{tg} \alpha_0 &= \\ &= \frac{(c_0^2 - c_3^2) [d - d \cos (n_2 \Delta\alpha_c) + c_2 \sin (n_2 \Delta\alpha_c)] - (c_0^2 - c_2^2) [d - d \cos \Delta\alpha_c + c_3 \sin \Delta\alpha_c]}{(c_0^2 - c_3^2) [c_0 - c_2 \cos (n_2 \Delta\alpha_c) - d \sin (n_2 \Delta\alpha_c)] - (c_0^2 - c_2^2) [c_0 - c_3 \cos \Delta\alpha_c - d \sin \Delta\alpha_c]} \end{aligned}$$

Equating the right sides of these equations will yield an equation for the single unknown $\Delta\alpha_c$. This equation is quite complex and cannot be solved by analytical methods. In case of necessity it can be solved graphically. From this, it is not difficult to determine the remaining parameters of the mechanism.

Here too it is necessary to remember, as mentioned in the previous example, that when assigning in advance specific ratios to the angles of rotation of the crank (corresponding to particular ranges of instrument readings) it may happen in certain cases, after solving the equations, that the initial angle α_0 or the final angle $\alpha_0 + \Delta\alpha_c$, and sometimes both these angles (if the i ratio has not been predetermined) are so large that the mechanism is either completely unable to function or

functions poorly. If this happens, it will be impossible to meet the requirements made on the mechanism in the given instance. Therefore, these demands must be reduced somewhat. For example, if the most uniform dial possible is desired, it will be necessary to settle for a slightly greater irregularity. With respect to instruments in which such a downgrading of the specifications is not acceptable (as, for example, if a uniform dial is absolutely essential), the derivation of inapplicable solutions shows that the proposed design of the transmitting mechanism is not suited to the instrument in question, and another linkage must be used instead.

In certain other amplifying linkages (particularly for complex mechanisms consisting of a number of linkages, each having a variable transmission ratio), the compilation and solution of equations analogous to those derived in the above examples, is quite complex, so that other methods of deciding on the linkage must be used. Nevertheless, in a number of cases this method of selecting the transmitting mechanism by simultaneous solution of a system of equations is applicable.

Let us also note that this method is sometimes used for other purposes. For example, in certain designs the transmitting mechanism is required not only to increase the travel of the pointer (or stylus) of the instrument relative to the travel of the pickup element, and not only to yield a dial of the required nature, but to perform also certain other functions. Thus, sometimes it is required of certain recording instruments (if, for example, the instrument has a vertical dial) that the tip of the pointer travel in a straight line. Several amplifying linkages have been developed to meet this requirement, of which the best known are the following:

1. Indicating element moving in guides and thus acquiring motion only from a link.
2. The so-called "Evans rocker" (Fig. 83).

Here the drive lever OA (having its axis at the point O) is hinged at the point A to an equal-arm lever CB (i.e. $CA = AB = OA$). The C end of the lever CB is the pointer (i.e., a pen inscribing the readings of a recording instrument may be

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attached at the point C). The point B can travel only along the straight line governed by the guide DD_1 , with the point O lying on the extension of this line. It is quite obvious that, in this arrangement, the point C is able to move only along the straight line OC.

3. Chebyshev's four-bar-motion mechanism (Fig. 84).

This linkage is based on the following hookup. Levers AC and BD are free to rotate about the axes A and B, which are fixed relative to the base of the instrument. To the ends C and D of these levers, the part DCE is hinged (the axes of the hinges D and C are parallel to the axes A and B). Point E of the part DCE is the pointer of the instrument. An appropriate determination of the dimensions of the components of this transmission yields the result that the point E will move virtually along the required path when the lever AC (or BD) rotates. This last type of linkage is the

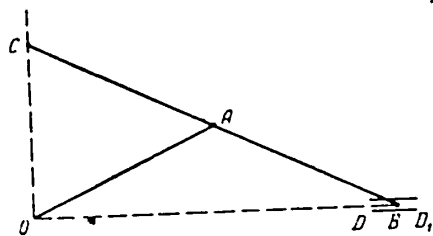


Fig. 83 - Diagram of "Evans' Rocker"

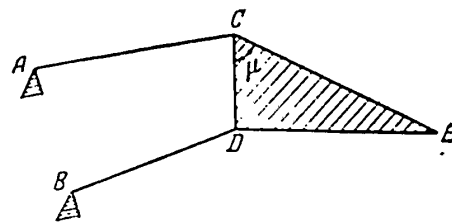


Fig. 84 - Diagram of Chebyshev's Four-Bar-Motion Mechanism

most desirable in many situations.

The requirement that the pointer of the instrument move along a fixed path creates additional conditions in the choice of parameters for the mechanism; therefore, the total sum of these additional conditions and of the conditions due to the required nature of the instrument dial, must not exceed the number of parameters selected in an analysis of the transmission by interpolation. A specimen determination of a linkage of this type is presented in the author's article on the selection of parameters for Chebyshev's unbalanced linkage (Bibl. 10).

CHAPTER X

HAIRSPRINGS IN INSTRUMENTS AND METHODS FOR THEIR CALCULATION

As mentioned above, in the majority of instruments equipped with mechanisms for transmitting movement from a pickup element to a pointer (indicator), a small auxiliary spring is mounted to the last component in the linkage (usually to the axis of the pointer). This spring produces tension in the entire mechanism in a single direction so as to provide an unbroken kinematic connection between all parts. Such a spring is known as a recoil or hairspring.

Note: Instruments without amplifying mechanisms, such as galvanometers, magnetic tachometers, thermometers with bimetallic spiral feelers in which the end of the spiral is attached directly to the axis of the pointer, do not have hairsprings, since their purpose is solely that indicated above.

In manufacturing practice it is common for hairsprings to be selected by the trial-and-error method, but it is true nonetheless that calculation permits a predetermination of the hairspring required by a given instrument. In the process of calculation, the starting point is the requirement to be met by the hairspring in actual service.

The first requirement to be met by a hairspring is that it be able to overcome all friction forces in the instrument mechanism. The displacements of the transmitting mechanism cause some work, due to friction in the mechanism. Since the hair-STAT spring must be able to move the entire amplifying mechanism, it must obviously be able to do not less work than the work lost to friction in the mechanism.

In addition, accelerations of the instrument casing may sometimes reach considerable magnitudes. These accelerations may result from accelerations of the entire body in which the instrument is mounted (an aircraft, for instance), and also from vibrations of the point at which the instrument is mounted, in which case the vibration may be excessive. Various parts of the mechanism may be incompletely balanced when this occurs. Therefore, in the presence of accelerations, certain parts of the mechanism will tend to undergo displacements relative to others, which may destroy the uniform constant kinematic linkage among all parts of the mechanism. The hairspring provides enough tension in the mechanism to prevent such interruptions. This may be considered a second requirement for the hairspring function.

In terms of the first of these requirements, the minimum moment M_0 of the hairspring, at the minimum angle of twist that it may undergo in the assembled instrument (corresponding either to the minimum or maximum reading) must be sufficiently larger than the moment of friction of the entire mechanism, reduced to the hairspring axis. This reserve is necessary since the forces of friction are variable and may change in magnitude for a great number of reasons: dirt, thickening of the lubricant, variation in temperature, accelerations of the attachment point of the instrument, etc.

Thus, we may write

$$M_0 = S\varphi_0 = k \left[\sum \left(F_i \frac{dy_i}{d\varphi} \right) + \sum \left(M_n \frac{d\alpha_n}{d\varphi} \right) \right], \quad (10.1)$$

where S is the stiffness of the hairspring (in units of moment per radian);

φ_0 is the angle of minimum tension of the spring (when its moment is M_0);

$\sum \left(F_i \frac{dy_i}{d\varphi} \right)$ is the total reduced friction force at those points in the mechanism where the point of contact between two parts moves progressively along some part of the instrument. Here F_i is the friction force resulting at some point in the mechanism comparable to i ;

dy_i is the simple displacement of the contact point, corresponding to the simple

rotation $d\varphi$ of the hairspring axis;

$\sum (M_n \frac{d n}{d\varphi})$ is the total reduced moment of friction in the axes of the mechanism (the axes of rotating parts, hinged joints, etc.). Here M_n is the moment of friction at some axis corresponding to n ; dn is the simple rotation of such a joint corresponding to the simple rotation $d\varphi$ due to the hairspring;

k is the assurance factor we referred to above. Usually k is set at several times unity, but since the hairspring, due to its pressure on the components of the mechanism, increases the friction forces therein, it is necessary, in determining the magnitude of the assurance factor k , to deal with the "efficiency" of the pickup of the instrument*, and in general with the degree of harmfulness of friction in the transmitting mechanism to the given instrument. If the instrument is "rugged" enough (i.e., if its pickup develops adequate forces or undergoes considerable displacement), there is no need to fear use of a large safety factor k . However, if the "efficiency" of the instrument pickup is small (if, for example, the instrument is used for measuring small forces, as in a climb indicator) or if, in the given instrument, the slightest increase in friction plays a significant role, it is desirable to avoid the use of too large an assurance factor k .

The friction force is considered proportional to the pressure between osculating parts. In our case, the pressure at points in the mechanism where friction is present results both from the weight of the parts themselves and from the pressure of the parts on each other, due to the hairspring proper. In the general case, for each individual part, the force of its own weight and the compression exerted by the hairspring differ in direction; therefore, in order for the total pressure to be attained

~~We~~ use this term to denote the product of the range of variation in the force developed by the instrument pickup at the attachment point of the transmitting mechanism and the total displacement of this point from the minimum to the maximum reading of the instrument. STAT

these forces must be added geometrically. This method may be excessively complex, however, or even impossible to realize. In fact, the magnitude of the given geometric sum varies in accordance with the position of the instrument (differently for each part of the mechanism), since the force of gravity varies its direction relative to the instrument in various positions of the latter, and the compressive force due to the hairspring action does not change in direction.

In view of the foregoing, we will, to simplify matters, always use the arithmetic sum of the forces instead of their geometric sum. This will slightly increase the value of the reduced force, but will merely serve as a sort of additional assurance factor (but not a very large one, as shown in the examples given below). On this assumption, the total reduced moment of friction M_{pr} [the term in brackets in eq.(10.1)] may be resolved into two components: 1) moment M_a , constituting the reduced moment of friction due to the weight of the parts, and 2) moment M_B , constituting the reduced moment of friction due to the hairspring effect. Thus,

$$M_0 \approx k[M_a + M_B].$$

The moment M_B is obviously proportional to the moment M_0 of the hairspring; i.e.,

$$M_0 \approx k(M_a + \xi M_B),$$

where ξ is a proportionality factor.

From this,

$$M_0 \approx \frac{M_a}{\frac{1}{k} - \xi} \quad (10.2)$$

The factor ξ is equal to $1 - \eta$, where η is the efficiency factor of the mechanism. In applied mechanics, it is usually considered that when a force is transmitted by means of some mechanism, only that portion of the force represented in our

equation by the term ξM_0 is lost to friction in the mechanism. The friction due to the weight of the parts is usually disregarded. This is explained by the fact that the forces acting on the mechanism are generally considered so large that the weight of the part can be neglected. However, in the case under examination, the moment of the hairspring is small so that virtually the total moment M_a is larger than the moment ξM_0 .

The moment M_a of friction, due to the weight of the parts, may be calculated in the following manner: Computing the weight of any part, the magnitude of the moment of friction in the axis of this part on rotation is determined (or the force of friction when the part is displaced along its guide). The resultant moment (or force) is reduced to the axis of the hairspring in the above manner (by multiplication by the ratio of the velocities of displacement of the part to the rotation of the axis of the hairspring). All such reduced moments are totaled. Of course, more accurate results would be obtained if each individual reduced moment were further divided into the efficiency of the part of the transmitting mechanism located between the part under examination and the axis of the hairspring; however, in our approximated calculation this may be neglected since it makes no great difference in practice.

If we do not know the efficiency ξ of the mechanism, the approximate value of the factor ξ can be calculated as follows: Having assigned to the hairspring some moment M_1 , we can calculate the pressure exerted by this moment on the axis and joints of the mechanism; for the sake of simplicity we determine these pressures in first approximation without, at this point, considering the friction in the mechanism, simply employing the usual methods of statics. This introduces only a very small error, since for a more exact calculation it would only be necessary to multiply the pressure calculated in this manner at some point by the efficiency of the portion of the mechanism between the axis of the hairspring and the point under examination. However, the efficiencies of our mechanisms are very close to unity. Therefore, if we ignore this factor in certain terms (usually in the smaller terms)

that add up to a given total, we increase the total by a very minor inaccuracy. In calculating the friction, considerably greater inaccuracies result from other causes, such as variability in the coefficient of friction, among others.

Determination of the pressure produced by the hairspring is preferably started from the axis of the spring itself. Let us take, as an example, a diagram for the joints of the parts close to the pointer as is encountered in the majority of instruments. The hairspring is mounted on the axis of the pointer O, and a small gear called the pinion and meshing with the rack is mounted to this axis. Thus, the following forces are at work on the part of the mechanism carrying the pinion: the compressive force P of the teeth of the rack and pinion, the reactive force of the bearings of the pointer axis O, and, in addition, the action of the hairspring in whatever fashion it is brought to bear. Both ends of the hairspring are fixed, one to some stationary part of the instrument and the other to a shoe seated on the axis. Theoretically, this type of spring (one with both ends fixed), in the ideal case, on rotation of the axis O, creates only a moment about this axis O and does not set up any other forces at the point of connection between hairspring and pinion, although in practice some forces will always be at work, however small.

Thus, if we retain the theoretical approach, the pinion is affected by the moment of the hairspring M_1 , the force P, and the reactive force in the pointer axis O. Under such conditions, the body can be in equilibrium only when the force P and the reactive force in the axis O constitute a couple, whose moment is equal to the moment M_1 (but in the opposite direction). Therefore, the compressive force induced by the hairspring in its own spindle is theoretically equal to the compressive force P of the pinion teeth on the teeth of the rack. In practice, this force will be somewhat larger than P. Therefore, the pressure on the pinion axis calculated in this manner may be increased by several percent (10-15%), but this may also be dropped since at several points in our calculation of the hairspring we have provided an adequate reserve, so that a slight reduction in a minor term will not be of

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major significance.

The force P is determined from the equation

$$Pr \cos \beta = M_1,$$

where r is the radius of the addendum (working) circle of the pinion;

β is the angle of mesh between the rack and the pinion.

In instruments it is desirable to use evolvent meshing, where the angle of mesh β is always constant. This angle is small (about $10-15^\circ$). At angles of this size, it is known that $\cos \beta$ does not differ greatly from unity; therefore, if the angle of mesh is unknown for some reason, the factor for $\cos \beta$ can be neglected and we may simply consider that $P = \frac{M_1}{r}$. In cases of cycloidal gear systems, the angle of mesh β is variable, but still does not reach an appreciable value; therefore, for the sake of simplicity (since this calculation is always approximated), the $\cos \beta$ factor is usually also ignored in the given equation (but P may, of course, be calculated for the worst case, i.e., the maximum angle β of the given gearing).

Knowing the compressive force P of the pinion teeth on the teeth of the rack, it is easy, by means of the most elementary methods of statics, to determine the reactive forces in all joints and shafts of the mechanism, due to the force in question. Knowing these forces, it is possible to determine the friction forces and moments resulting therefrom at various points in the mechanism. All these forces and moments of friction may, by means of the methods known to us, be reduced to the axis of the hairspring (by multiplication by the ratio of the rate of displacement of the given part to that of the hairspring axis). Addition of all these reduced moments yields a moment M_2 . Division of this moment by the moment M_1 , which we had assigned, will obviously yield our coefficient ϵ .

If, in the operation of an instrument, considerable accelerations are possible, including vertical accelerations (termed "overloading") in aircraft instruments, and proper functioning of the given instrument in the presence of such accelerations is

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desired, then the value of M_a in eq.(10.2) must be multiplied by n ; this number indicates the number of times by which the geometric sum of the accelerations of the installation point of the instrument and the acceleration of gravity (considering this acceleration to be directed upward) are greater than the acceleration of gravity g alone. It is true, however, that if the number n is considerable, a smaller assurance factor k may be used (sometimes not much greater than unity), due to the fact that, for the majority of instruments in which such large accelerations are possible, the maximum accelerations occur quite rarely.

However, in all the equations derived above, it is only the friction forces in the mechanism that are provided for. Yet one must remember that the parts of the mechanism cannot always be considered to be in a state of ideal balance. Therefore, if we do not even consider operation of the instrument under conditions of acceleration, eq.(10.2) must be extended somewhat, as follows:

$$M_0 = \frac{M_a}{\frac{1}{k} - \xi} \pm M_{unb}, \quad (10.3)$$

where M_{unb} is the overall total moment of unbalance of the mechanism, reduced to the hairspring axis, from that axis to a point at which disruption of the kinematic connection in the mechanism might occur (the calculation of this moment has been discussed above in the paragraph on methods of calculating certain instrument errors).

If the second of the above requirements of hairspring operation must be satisfied, i.e., the requirement that the kinematic connection of the parts of the mechanism remain intact under acceleration of the instrument, it is obviously necessary that in eq.(10.3) M_{unb} be multiplied by that same number n , by which the moment M_a has to be multiplied in this case, so that we will have

$$M_0 \geq \frac{nM_a}{\frac{1}{k} - \xi} \pm nM_{unb}. \quad \text{STAT} \quad (10.4)$$

However, this equation pertains to a case involving an acceleration acting in a

single direction for a reasonable period of time. However, as indicated above, if the instrument has to operate under vibration of the part to which it is mounted (as in an aircraft), allowance for this factor is quite important. In this case, the condition satisfying this second requirement of hairspring operation must be derived somewhat differently.

Let us examine this problem for the simplest case, in which vibration occurs in accordance with the laws of harmonics.

Let us take some contact point A between two parts of the mechanism, in which the determinate kinematic connection may be disrupted.

To prevent this rupture, the compressive force between two adjacent parts must always be positive.

Let us begin by writing a general equation of motion for the mechanism, vibrating at the installation point of the instrument, while assuming a stable kinematic connection between the parts. In this case (ignoring friction in the mechanism), we have

$$m_{np}\ddot{x} + B\dot{x} + cx = \frac{(P_{npr})_x}{g} a\omega^2 \sin \omega t, \quad (10.5)$$

where x is the displacement from its mean position, as a result of vibration, of the point to which all forces are reduced (i.e., the point at which the linkage is joined to the pickup);

m_{pr} is the reduced mass of the entire mechanism,

B is the damping coefficient, if any (in the present calculation, if the damping of the instrument is small, it may be assumed that $B = 0$ for purposes of assurance);

c is the total rigidity of the system (thus, if the reduced rigidity of the hairspring is not very small relative to the rigidity of the elastic piSTAT then it is necessary to supplement the rigidity of this pickup by the approximate value of the reduced rigidity of the hairspring; but in a number

of cases this reduced rigidity may be ignored);

$(P_{n.pr})_x$ is the total reduced force of unbalance of all parts of the mechanism that would result if the force of gravity were operative in the direction of vibration;

a is the amplitude of vibration;

ω is the frequency of vibration.

From this we obtain

$$x = \frac{(P_{n.pr})_x}{m_{pr}g} \lambda_1 a \sin(\omega t - \delta),$$

where λ_1 is the dynamic receptivity factor of the system during vibration;

δ is the phase shift (the equation for these values has been derived previously).

In this case, the angles of rotation φ of the hairspring axis from its mean position as a result of oscillatory motion about that axis are

$$\varphi = \left(\frac{d\varphi}{dx} \right) \frac{(P_{n.pr})_x}{m_{pr}g} \lambda_1 a \sin(\omega t - \delta), \quad (10.6)$$

while the transmission ratio $\frac{dx}{d\varphi}$ from the pickup to the hairspring spindle is taken as constant. This is permissible in the given calculation, since the movement of the mechanism on vibration is usually quite insignificant.

If we now take only the portion of the mechanism from the hairspring axis to the point under examination A, in which we seek to eliminate the possibility of disruption of the specific kinematic connection between the parts, the equation of motion for this part of the mechanism will be (reducing all forces and moments to the hairspring axis).

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$$(J_{pr})_s \ddot{\varphi} + S\varphi + S\varphi_n - M_T = \frac{(M)_s}{g} a \omega^2 \sin \omega t, \quad (10.7)$$

where $(J_{pr})_B$ is the moment of inertia, reduced to the hairspring axis; of the entire mechanism from the hairspring spindle to the point A;

S is the stiffness of the hairspring;

φ_H is the angle of winding of the hairspring when the axis is in the center position during vibration;

M_T is the reduced moment due to the compressive force T of the parts in contact at the point A;

$(M_{n.pr})_B$ is the total moment of unbalance of the entire mechanism, between the hairspring axis and the point A, that would be obtained if the force of gravity had acted in the direction of vibration.

From this equation we obtain

$$M_T = S\varphi_H - \frac{(M_{n.pr})_B}{g} a\omega^2 \sin \omega t + S\varphi + (J_{pr})_B \ddot{\varphi}.$$

If the kinematic connection between the parts of the mechanism is not disrupted by vibration, the value of the angles φ in eq.(10.7) will not be determined from this equation but from the general equation (10.5), i.e., it will be equal to expression (10.6). Therefore,

$$\begin{aligned} M_T &= S\varphi_H - \frac{(M_{n.pr})_B}{g} a\omega^2 \sin \omega t + \\ &+ \left(\frac{d\varphi}{dx} \right) \left[S - (J_{pr})_B \omega^2 \right] \frac{(P_{n.pr})_x}{m_{pr}g} \lambda_1 a \sin (\omega t - \delta) = \\ &= S\varphi_H - a \left\{ \frac{(M_{n.pr})_B}{g} \omega^2 \sin \omega t + \right. \\ &+ \frac{d\varphi}{dx} \left[(J_{pr})_B \omega^2 - S \right] \frac{(P_{n.pr})_x}{m_{pr}g} \lambda_1 \cos \delta \sin \omega t - \\ &\left. - \frac{d\varphi}{dx} \left[(J_{pr})_B \omega^2 - S \right] \frac{(P_{n.pr})_x}{m_{pr}g} \lambda_1 \sin \delta \cos \omega t \right\} \end{aligned}$$

or, if we introduce simpler symbols,

$$\frac{(M_{n.pr})_B}{g} \omega^2 = D;$$

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$$\frac{d\varphi}{dx} [(J_{pr})_u \omega^2 - S] \frac{(P_{n.pr})_x}{m_{pr}g} \lambda_1 = E, \quad (10.8)$$

we obtain, after simple transformations,

$$M_T = S\varphi_H - a \sqrt{E^2 + D^2 + 2ED \cos \delta} \sin(\omega t - \delta_1), \quad (10.9)$$

where

$$\delta_1 = \arctg \left(\frac{E \sin \delta}{D + E \cos \delta} \right).$$

Thus, in order to have eq.(10.9) always yield a positive value for the force T (with the result that the kinematic connection at this point will not be interrupted), it is necessary that

$$S\varphi_H > |a \sqrt{D^2 + E^2 + 2ED \cos \delta}|, \quad (10.10)$$

where D and E are the expressions presented in eq.(10.8).

In view of the fact that the mean angle φ_H of winding of the hairspring during vibration is virtually identical with the same angle in the absence of vibration (the slight discrepancy in these angles, due to the Maxwell effect, is usually infinitesimal) and in view of the fact that the condition written above must be satisfied even at a minimum angle of winding φ_0 , then, obviously, we may substitute φ_0 for φ_H in the inequality (10.10); this will yield another new condition for limiting the minimum value of the angle φ_0 , in addition to the above condition (10.10).

For example, let the force of unbalance $(P_{n.pr})_x$ of the entire mechanism (when the kinematic connection in the connector link hinges has not yet been disrupted) reduced to the pickup element, be about +1 gram, and let the quantity $(M_{n.pr})_B$ (to which the total reduced mass of the balancing load on the pickup is added since, at the instant of disruption of the kinematic connection in the link hinges this load is not counterbalanced) be approximately 0.3 gm-cm; $(J_{pr})_B \approx 0.001 \text{ gm-cm-sec}^2$; STAT

$m_{prg} \approx 500 \text{ gm}$; $\frac{d\varphi}{dn} \approx 30 \text{ rad/cm}$. Let the vibration of the attachment point of the instrument have a frequency of 200 sec^{-1} , and an amplitude of 0.5 mm . Also, let us assume that, in the preliminary solution of eq.(10.5) for the given mechanism and its parameters of vibration, we obtained the quantity λ_1 as approximately equal to 1.3 , at an angle δ of 160° . Then, for a hairspring with a stiffness of 0.5 gm-cm/rad , we obtain

$$D \approx \frac{0.3 \cdot 200^2}{981} \approx 12.24 \text{ g};$$

$$E \approx 30 [0.001 \cdot 200^2 - 0.5] \cdot \frac{1}{500} \cdot 1.3 \approx 3.1187 \text{ g}.$$

Since, in the given example, $\cos \delta < 0$, we will take the worst case in which

$$(P_{n.pr})_x = -1 \text{ gm}.$$

Then,

$$\sqrt{D^2 + E^2 + 2DE \cos \delta} \approx \sqrt{12.24^2 + 3.1187^2 + 2 \cdot 12.24 \cdot 3.1187 \cdot 0.9397} \approx 15.209 \text{ g}.$$

Therefore,

$$0.5 \varphi_0 > 0.05 \cdot 15.209,$$

and this means that the angle φ_0 must be greater than $\frac{15.209}{10} \approx 1.5209$, which is virtually equal to $\frac{\pi}{2}$.

Note: In the derivation of the conditions (10.10) and the expressions E and D , we allowed for damping only in eq.(10.5) for the oscillations of the entire mechanism and did not allow for damping in eq.(10.7) for the oscillations of the portion of the mechanism from the hairspring shaft to the point of possible disruption of the constant kinetic connection in the mechanism. ^{STAT}
We also permitted an approximation to obtain primarily some degree of assurance (the optimum design for hairsprings calls for a factor of safety), since

damping in this portion of the mechanism may be insignificant, and secondly to obtain eq.(10.10) in a simplified form. If we had also allowed for some damping in eq.(10.7) as well, we would have emerged with a condition similar in structure to eq.(10.10), but somewhat more complicated. In this connection, we note that such damping also facilitates a more ready satisfaction of condition (10.10).

If the balancing of all parts of the mechanism is sufficiently accurate, the result in many designs will be that, with progressive accelerations, the absolute magnitude of M_{unb} will not exceed the magnitude of the moment M_a , i.e.,

$$M_{unb} < M_a,$$

as the accuracy of the usual balancing of a part is limited by the moment of friction M_{Tp} of this part relative to its shaft, i.e.,

$$0 < |M_{unb.p}| < M_{pr},$$

where $M_{unb.p}$ is the residual moment of unbalance of the part after balancing.

It is impossible to contend that all the parts of a mechanism will undergo an absolute maximum of inaccuracy of balance and, in addition, the moments of unbalance of individual parts may act in opposite directions. Therefore, it must be assumed that when the balancing is satisfactory, the total M_{unb} will be somewhat less than M_a ; however, in a number of cases an inadequately accurate balancing of the parts of mechanisms in instruments occurs. For example, sometimes the counterweights of parts may be standardized and identical for all instruments of a series, without verification of whether the part is properly balanced in each specific mechanism. As a result, considerable unbalance in some part of a mechanism is sometimes observed. This may have the result that the moment M_{unb} in expression (10.4) will ^{not} be smaller but larger than M_a , and sometimes even considerably larger. In this case, obviously, there is a corresponding increase in the magnitude of

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 $a\sqrt{D^2 + E^2 + 2ED \cos \delta}$, limiting the minimum permissible magnitude of M_0 in accordance with condition (10.10).

Therefore, in making use of conditions (10.4) and (10.10) we must know in advance the approximate degree of accuracy of balance of the parts we can expect for the given series of instruments.

In addition, in certain designs, the magnitude of the reduced moment $(M_{n.pr})_B$ may, in individual cases, be quite large despite an entirely accurate balancing of the parts of the mechanism. We have already encountered such a case in our discussion of the inadequacies of balancing the pickups on the basis of the diagram in Fig.27.

Further, even when the balancing of a pickup element is more complete, in accordance with the diagram presented in Fig.26, it is necessary to take into account the possibility of violation of the determinate kinematic connection in the hinges A and B of the connector link, since there is always some clearance in these hinges; in addition, in the analysis in question, the expression $(M_{n.pr})_B$ also contains the term $qb \frac{d\psi}{d\varphi}$, where q is the weight of the counterpoise, b is the length of the arm OD, and $\frac{d\psi}{d\varphi}$ is the ratio of the velocities of rotation of the crank OA and of the axis of the hairspring (first noted, apparently, by engineer V.A.Shcherbakov).

Attention should also be drawn to the possibility of disruption of the determinate kinematic connection between parts of the mechanism if the instrument is subject to angular vibrations or to any type of angular accelerations. True, angular vibrations are usually quite small, but it must be remembered here that balancing in the sense of compensation of angular accelerations is usually not provided for in instrument mechanisms.

For angular accelerations, the term $n\ddot{\varphi}_{unb}$ in eq.(10.4) must be replaced by $M\ddot{\varphi}$ which is equal to

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$$M\ddot{\varphi} = \sum \left(J_n \cos \beta_n \frac{da_n}{d\varphi} \right),$$

where $\ddot{\gamma}$ is the magnitude of the possible angular acceleration in the body of the instrument;

J_n is the moment of inertia of a part n , able to rotate about its axis;

β_n is the angle between the axis of rotation of the part and the axis about which angular acceleration $\ddot{\gamma}$ occurs;

$\frac{d\alpha_n}{d\varphi}$ is the ratio of the angular velocity of rotation of the part under examination to the velocity of rotation of the hairspring shaft. The given sum contains moments of all parts rotatable about some axis. Therefore, for this case, eq. (10.4) is rewritten in the form

$$S\ddot{\varphi}_0 > \frac{M_a}{\frac{1}{k} - \xi} + \ddot{\gamma} \sum \left(J_n \cos \beta_n \frac{d\alpha_n}{d\varphi} \right). \quad (10.4a)$$

Proceeding to a case of angular vibration, let us again examine the simplest case of harmonic vibration, i.e., let us consider

$$\gamma = \gamma_0 \cdot \sin \omega t,$$

where γ is the angle of rotation of the body of the instrument on vibration;

γ_0 is the possible angular amplitude of vibration.

Then, in expressions aE and aD, representing terms in condition (10.10), the quantities $\frac{(P_{n.pr})x^a}{g}$ and $\frac{(I'_{n.pr})B^a}{g}$ are replaced, respectively, by the terms

$$\gamma_0 \sum \left(J_n \cos \beta_n \frac{d\alpha_n}{dx} \right) \text{ and } \gamma_0 \sum \left(J_n \cos \beta_n \frac{d\alpha_n}{d\varphi} \right)$$

and in place of condition (10.10) we will have the condition

$$S\ddot{\varphi}_0 > \left| \gamma_0 \sqrt{D_1^2 + E_1^2 + 2D_1E_1 \cos \delta} \right|, \quad (10.10a)$$

where

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$$D_1 = \omega^2 \sum \left(J_n \cos \beta_n \frac{d\alpha_n}{d\varphi} \right);$$

$$E_1 = \frac{d\varphi}{dx} \frac{[(J_{pr})_n \omega^2 - S]}{(m_{pr})_x} \lambda_1 \sum \left(J_n \cos \beta_n \frac{da_n}{dx} \right).$$

Normally, however, eq.(10.10a) is not used in the design and calculation of instruments, in view of the insignificant magnitude of the angular vibrations, and also because the amplitudes of these vibrations are unknown to us for the most part. For similar reasons, eq.(10.4a) is also not employed.

The minimum moment M_0 of the hairspring, determined by means of eqs.(10.4) and (10.10), is equal to the product $S\varphi_0$, where φ_0 is the minimum angle of twist of the hairspring. The maximum moment M_{max} of the hairspring is $S(\varphi_0 + \varphi_p)$, where φ_p is the total angle of rotation of the pointer axis from the minimum to the maximum reading of the instrument.

Thus,

$$M_{max} = M_0 \frac{\varphi_0 + \varphi_p}{\varphi_0}.$$

From this it is clear that it would be better if the $\frac{\varphi_p}{\varphi_0}$ ratio were as small as possible since, in that case, M_{max} is at its smallest. This is advantageous in view of the fact that the hairspring moment also increases the friction in the mechanism. However, in the majority of instruments, φ_p is virtually equal to the complete arc of the pointer. In addition, hairsprings usually function properly until subjected to an angle of twist somewhat larger than 1.5 rotations. Therefore, it is generally impossible to use a very large φ_0 angle. Usually, this is equal to about a quarter revolution ($\varphi_0 = \frac{\pi}{2}$). Therefore, M_{max} is usually 4.5 or even 6 times as large as M_0 (in cases in which φ_p is greater than a single revolution).

Having assigned the magnitude of φ_0 on the basis of this or other considerations and knowing the magnitude of M_0 , we determine the required elasticity S of the hairspring. Knowing the magnitude of the maximum moment of the hairspring M_{max} and its elasticity S , we use the standard formulas for spiral springs to determine the

dimensions of the hairspring, i.e., its thickness, width, and developed length (the sum of all turns).

It is quite obvious that a hairspring, in addition to overcoming friction in the transmitting mechanism, also exerts some effect on the pickup. In general, we have a system here in which there are two elastic elements mounted at different points in the system: the mainspring of the instrument and the hairspring. Therefore, strictly speaking, it would be necessary, in calculating the travel of the pickup, to reduce the elasticity of the hairspring to the point of application of the force of the mainspring, i.e., instead of the elasticity of the spring element alone it would be necessary to use the sum of the elasticity of this element and the reduced elasticity of the hairspring.

As we know, this reduced elasticity of the hairspring is approximately $S\left(\frac{d\phi}{dx}\right)^2$. But the elasticity of the hairspring is quite small in most instruments, and therefore its reduced elasticity in many instruments is insignificant by comparison with the elasticity of the main elastic element, despite the fact that the axis of the hairspring rotates through a fairly large angle ϕ , at an insignificant displacement x of the point of attachment of the transmission mechanism to the pickup (the square of the $\frac{d\phi}{dx}$ ratio being taken when this reduction is performed). Therefore, in many instruments it is entirely legitimate to ignore the elasticity of the hairspring in calculating the pickup element. But in very "weak" instruments, such as rate-of-climb indicators, it is often necessary to consider this factor.

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CHAPTER XI

TEMPERATURE COMPENSATION OF INSTRUMENTS

Section 1. Bimetal "Kinematic" Temperature Compensation of Instruments

We know that a temperature error exists in many instruments. In the case at issue we are not discussing instruments which are less exact at certain temperatures, due to increase in friction in the mechanism and other similar causes. What we have in mind here are instruments that show some variance in reading with changes in temperature (although the readings themselves may be entirely accurate and determinate for each particular temperature). Such variations in instrument readings, due to temperature, may be caused by different factors in different instruments. Thus, in instruments whose design provides for an elastic element, the major cause of temperature errors in the majority of cases is a variation in the modulus of elasticity of the material of the elastic element on variations in temperature. In electric and magnetic instruments, the following factors may affect the readings of an instrument during temperature fluctuations: variation in magnetic permeability, electric resistance, self-induction, amount of clearance between parts, etc.

Changes in instrument readings with changes in temperature naturally cause difficulties in their use. Therefore, instruments are often provided with what is known as temperature compensation, i.e., design elements are introduced, also subject to the effect of temperature, and therefore also serving to vary the instrument reading on temperature fluctuations. This compensation is determined so that the change in instrument readings, due to compensation, is opposite and if possible equal

in absolute magnitude to the variation in the indications observed in the instrument in the absence of compensation.

Various methods of compensation exist. For example, in aneroid cells, a certain amount of gas is sometimes introduced for this purpose (although this method is rarely used in view of certain of its disadvantages). In magnetic and electric instruments, the structural materials of various parts are selected in accordance with the foregoing, shunts made of special alloys are employed, etc. But many of these methods have limited application in the analysis of special instruments, so that they will be disregarded in this publication. Certain of these, however, have come into sufficiently widespread use (i.e., temperature shunts in electric instruments), but they fall completely into the field of electrical engineering and have been discussed in adequate detail in the appropriate sources.

Of the mechanical methods of thermal compensation, the most widely used is that of bimetal strips inserted into the transmitting mechanism of the instrument. This method is the most universal since many instruments have amplifying mechanisms, and it is almost always possible to find place therein for these particular types of compensation.

This type of compensation is termed "kinematic" bimetal thermal compensation, as distinct from the so-called bimetal "force" compensation that has recently made its appearance, in which the bimetal strip presses with some force directly on the pickup element of the instrument.

In this Chapter, by way of example, we will examine the procedure for calculating the most frequently encountered systems of kinematic bimetal compensation, to wit, that in diaphragm instruments having crank couplings in their transmitting mechanisms.

Such instruments usually have bimetal temperature compensations in two places. One, termed primary compensation, is in the pickup element itself; and the other, termed secondary compensation, is on the crank of the transmitting mechanism des-

cribed*. These two compensations affect the instruments differently, due to their arrangement.

In secondary compensation, as already indicated in the Chapter on bimetal strips, the strips are so placed that the length of the crank changes with their deflections (see Figs. 67 and 65), since the length of the crank in this case is the distance from the axis of rotation of the shaft (2) of the B end of the strip, which is connected to the link by a hinge. Thus, the temperature deflections of the strip affect the transmission ratio of the mechanism.

We know that an increase or decrease in the length of this crank results in an overall decrease or increase in the transmission ratio of the mechanism. However, a variation in temperature results in a variation in the elasticity of the box, i.e., in the value of $\frac{dp}{dx}$, where $\bar{x}$ is the deflection of the box. A decrease in temperature induces an increase in $\frac{dp}{dx}$, and an increase in temperature causes a decline in this ratio. Therefore, in order to prevent a change in the value of $\frac{d\varphi}{dp}$, where φ is the angle of rotation of the pointer, it is necessary that, a reduction in temperature be accompanied by an increase in the amplification ratio $\frac{d\varphi}{dx}$, while a rise in temperature be accompanied by a reduction in this ratio. The purpose of secondary compensation is to change the transmission ratio of the mechanism in accordance with variations in the elasticity of the box.

It is obvious, however, that such compensation may prove inadequate in many cases. Actually, if we have two functions $\alpha_p = f_p(p)$ and $\alpha_t = f_t(p)$, where α_p is the angle of rotation of the crank at the temperature for which the calculation is made and constitutes a function $f_p(p)$ of the measured pressure p , while α_t is the angle of rotation of the crank at a somewhat different temperature t , also constituting a function $f_t(p)$ of the pressure p , and even if we have attained the result that

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*Secondary temperature compensation was first developed in the USSR by G.O. Fridlender, engineer (and apparently at an earlier date than abroad).

$$\frac{d\alpha_p}{dp} = \frac{d\alpha_t}{dp}, \quad (11.1)$$

this still does not mean that the equation

$$\alpha_p = \alpha_t. \quad (11.2)$$

will also be valid.

Equation (11.2) will obviously be valid only if, in addition to eq.(11.1), the following condition is satisfied. The original angle of the crank α_0 , corresponding to the initial reading of the instrument (i.e., to the initial pressure p , which may sometimes not be a zero reading, as for example in altimeters) must be identical both at the temperature for which the calculations are made and at a changed temperature. The requirement, therefore, is that

$$(\alpha_0)_p = (\alpha_0)_t.$$

However, there are a number of cases in which these angles may not agree. For example, if at the initial reading of the instrument some thermal variation in the deflection of the pickup element is already present (as in altimeters in which the entire pressure of the atmosphere is at work on the aneroid cell at its initial reading at sea level) and the angle between the crank and the connector link at this reading is $\frac{\pi}{2}$ (so that a slight variation in the length of the crank will not cause a rotation of the crank), then obviously the angle $(\alpha_0)_t$ will not equal the angle $(\alpha_0)_p$, if we limit ourselves to the use of only secondary compensation.

Identically, if at the initial reading of the instrument the pickup does not change its deflection due to variations in the temperature, but, at this initial reading, the angle between the hinge and the connector link is either acute or obtuse, a variation in the length of the crank will obviously induce some variation in the angle α_0 .

It follows from the foregoing that the condition (11.2) can be satisfied only if an additional compensation is introduced into the mechanism other than secondary compensation. As a result, such mechanisms are usually also provided with primary compensation which displaces the initial position of point B of the connector link (i.e., the point of connection between the transmitting mechanism and the pickup element) when the temperature changes. A combination of primary and secondary compensation must compensate for the greater part of the thermal error of the instrument at all readings.

Thus, primary compensation should alone compensate for thermal variations in the deflection of the pickup element not at the initial reading of the instrument, as stated in certain textbooks, but at that reading at which the transmitting mechanism is in such a position that the function of secondary compensation does not affect the position of the pointer. In calculating "kinematic" thermal compensation we designate this as the zero position of the mechanism, while this initial position of the mechanism fails in many cases to correspond to the initial reading of the instrument. For example, in the case of the crank mechanism under examination; this is the position of the mechanism at which the angle between connector link and crank is $\frac{\pi}{2}$. At all readings other than that cited, thermal compensation is obtained through the combined action of both compensations.

Let us take, for example, the crank mechanism of Fig.67. Here, OA is a crank whose length is denoted by a ; AB is a connector link whose length is denoted by b ; BD is the direction of motion of the pickup element; OC is a direction normal to BD; O_1 is the point of intersection of the straight lines BD and OC; α is the angle between the direction OC and the crank; δ is the angle between BD and the crank; γ is the angle between the crank and the connector link; the distance OC_1 will be denoted by d .

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As in the section on the major varieties of the components of transmitting mechanisms, we have

$$\left. \begin{aligned} O_1B &= b \cos \delta - a \sin \alpha; \\ \sin \delta &= \frac{a \cos \alpha - d}{b}; \\ \cos \delta &= \sqrt{1 - \left(\frac{a \cos \alpha - d}{b} \right)^2} \end{aligned} \right\} \quad (11.3)$$

and

$$O_1B = b \sqrt{1 - \left(\frac{a \cos \alpha - d}{b} \right)^2} - a \sin \alpha. \quad (11.4)$$

With a change in temperature, the length of O_1B varies by a value of $\Delta(O_1B)$, equal to

$$\Delta(O_1B) = \Delta x - \Delta c, \quad (11.5)$$

where Δx is the variation in the deflection of the pickup due to changes in temperature, which we consider positive in a direction opposite to the direction BD;

Δc is the deflection of primary compensation, resulting from the same change in temperature, which we consider positive in the same direction as BD.

This change in temperature also causes the crank a to vary in length by a magnitude Δa , due to the presence of secondary compensation.

If the thermal compensations, taken together, result in a correct compensation of the temperature error of the instrument, the angle α must remain unchanged with any change in temperature.

The values d and b are constant. Thus, in this case only the values O_1B and a vary in eq. (11.5). In accordance with the theorem of finite differences, we may write

$$\Delta(O_1B) = \left[\frac{d(O_1B)}{da} \right]_{a=a_{pr}} \Delta a, \quad (11.6)$$

where a_{cp} is a mean length of the crank a , between its length a_0 at the initial temperature and its length a_t at the changed temperature.

In view of the fact that the relative variations in the quantities O_1B and a are quite small in the given instance, we may write, approximately:

$$\Delta(O_1B) \approx \left[\frac{d(O_1B)}{da} \right]_{a=a_0} \Delta a. \quad (11.6a)$$

Substituting in this equation the expressions O_1B and $\Delta(O_1B)$ from eqs. (11.4) and (11.5), and obtaining the derivative thereof, we arrive at

$$\Delta x - \Delta c \approx \left[- \frac{\cos \alpha \left(\frac{a \cos \alpha - d}{b} \right)}{\sqrt{1 - \left(\frac{a \cos \alpha - d}{b} \right)^2}} - \sin \alpha \right] \Delta a.$$

Substituting, and using the equations of the system (11.3), we obtain

$$\Delta x - \Delta c \approx - \left[\frac{\cos \alpha \sin \delta}{\cos \delta} + \sin \alpha \right] \Delta a = - \frac{\sin(\alpha + \delta)}{\cos \delta} \Delta a = - \frac{\cos \gamma}{\cos \delta} \Delta a,$$

from which it follows that

$$\Delta c \cdot \cos \delta - \Delta a \cos \gamma \approx \Delta x \cos \delta. \quad (11.7)$$

The value Δc is the effect of primary compensation, while the value Δa is the effect of secondary thermal compensation. From this it is evident that the thermal error Δx is compensated in the general case by the combined effect of the two compensations. In this case, while the angle γ is larger than $\frac{\pi}{2}$, the coefficients of Δc and Δa in eq. (11.7) are identical in sign ($\cos \gamma < 0$). When $\gamma = \frac{\pi}{2}$, the coefficient of Δa is zero, and only primary compensation is at work, while when $\gamma < \frac{\pi}{2}$ the coefficients of Δc and Δa in eq. (11.7) differ in sign. STAT

It should be noted that this double temperature compensation does not, generally speaking, provide ideal compensation at all readings of the instrument. As dem-

onstrated below, theoretically it compensates completely for the thermal error only at two readings of the instrument. For other readings it may not compensate for this error with complete accuracy.

This is due to the fact that, in the given case we have only two parameters for improving the compensation. We may, for this purpose, only assign thermal deflections Δc and Δa of the two bimetal strips (selecting the corresponding dimensions thereof). This is also evident from eq.(11.7), containing only the two unknowns Δc and Δa . Therefore, applying here the usual method of interpolation for determining these parameters, we may, in order to obtain a specific solution of the problem, write only two equations of the type of eq.(11.7) for any two readings of the instrument.

The reason for this phenomenon is the following: Let the change Δa in the length a of the crank, for some reading of the instrument, induce such a change in the transmission ratio $\frac{dp}{dx}$ as to cause it to compensate the change in elasticity $\frac{dp}{dx}$ resulting from the given change in temperature. This being the case, it is entirely possible that, at other readings of the instrument due to the factors indicated, the variations in magnitudes $\frac{dp}{dx}$ and $\frac{dp}{dx}$ no longer compensate each other completely. In any case, here too there is a compensation of most of the thermal error, so that only a small portion thereof remains.

The main reason why variations in the magnitudes of $\frac{dp}{dx}$ and $\frac{dp}{dx}$, compensating each other at any two positions of the mechanism, do not compensate each other completely when the mechanism is in other positions, is the variability of $\frac{dp}{dx}$, i.e., the nonlinearity of the ratio between magnitudes p and x (which is found in a very large number of instruments). A cursory examination of this question reveals, moreover, that it would be more theoretically correct in general (but only in a somewhat different connection than the effect of variability in $\frac{dp}{dx}$ to which we STAT have just referred) to mount the secondary compensation on a driving and not a driven arm of the transmitting mechanism.

As already indicated (Sect.3, Chapter I), the formulas for the deflection of elastic elements under load take the following appearance: $P = E_0 f_1(x)$, or $P = G_0 f_2(x)$ when the load P is concentrated, and $p = E_0 \varphi(x)$ in a case in which the load p is distributed, where $f_1(x)$, $f_2(x)$, and $\varphi(x)$ are functions of the deflection x of an elastic element at the calculated initial temperature, and also the dimensions of the elastic element. When the temperature varies by Δt we may assume, roughly, that

$$p = E_0(1 - \beta \Delta t) \varphi(x)$$

(here, for our subsequent discussion, we will use only one of the listed types of formulas for the deflection of elastic elements, since in other cases the reasoning will be entirely analogous).

If the relationship between the values p and x is linear, we have

$$x_0 = \frac{p}{E_0 \psi},$$

where x_0 denotes the deflection of the elastic element at the calculated zero temperature, and ψ is a function of the dimensions of the elastic element. Further we have, approximately

$$x_{\Delta t} = \frac{p}{E_0(1 - \beta \Delta t) \psi},$$

where $x_{\Delta t}$ is the deflection of the elastic element at a temperature varying by Δt , but at the same load p as before.

In Chapter IX [compare eqs.(9.11a) and (9.11)] we showed that the displacement of the terminal point (or the rotation α of the final link, as for example the pencil) of the mechanism is related to the displacement x of the initial point in the mechanism by an equation that, in the majority of cases, has roughly the appearance: STAT

$$y = \frac{R_1}{R_2} N x,$$

where R_1 is the product of the lengths of the driving arms of the levers in the mechanism;

R_2 is the product of the lengths of the driven arms of the mechanism;

N is a function of the angles between meshing parts, functionally related to the displacements x and y .

Thus, we may write

$$y_0 = \frac{R_1}{R_2} \frac{N}{\psi} \frac{p}{E_0}$$

The length of the arm bearing the secondary compensation may be shown as

$$a + b\Delta t,$$

where a and b are constants dependent on the parameters of the device, and b may either be positive or negative depending on the manner in which the bimetal strip is mounted. Therefore, if secondary compensation is provided at some driving arm, then at a temperature varying by Δt , we will have the following value $y_{\Delta t}$ of displacement of the terminal link in the mechanism:

$$y_{\Delta t} = \frac{R_1 \left(1 + \frac{b}{a} \Delta t\right)}{R_2} \frac{N_1}{\psi} \frac{p}{E_0 (1 - \beta \Delta t)}$$

Here, instead of N we write expression N_1 , since a thermal variation in the length of the arm may, in many instances, cause some slight change in the angles between the parts of the mechanism (but of course, for the most part, this change in the function N is quite negligible and may therefore be ignored).

Thus, by selecting the parameters of compensation so that $\frac{b}{a} = -\beta$, and assuming $N_1 \approx N$, we obtain, roughly, in case of a linear ratio between p and x , approximately complete compensation at all readings of the instrument and at all variations in temperature Δt .

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Should the secondary compensation be installed on the driven arm, we have

$$y_{\Delta t} = \frac{R_1}{R_2} \frac{N_1}{\psi} \frac{p}{E_0 \left(1 + \frac{b}{a} \Delta t\right) (1 - \beta \Delta t)}$$

Here the magnitude $\frac{b}{a}$, required so that the product of $\left(1 + \frac{b}{a} \Delta t\right) (1 - \beta \Delta t)$ is unity, is a function of the value of Δt ; since, in an instrument already calibrated, the value of $\frac{b}{a}$ is constant, then strictly speaking, complete compensation at this instrument reading is obtained only at a specific value of Δt .

In relation to the derivations made above [see eqs.(11.6) and (11.6a)] and to certain other derivations [see eq.(11.11)], this is expressed in the fact that an approximated transition from differentials to finite differences (leaving the magnitudes of the derivatives unchanged in this respect) is more legitimate, i.e., it introduces smaller errors into the calculation in a case in which the secondary compensation is provided at the driving arm than when it is mounted on the driven arm.

If, however, we have a nonlinear relationship between the quantities p and x , then, for practical purposes, the inaccuracy to which we have referred at the driven arm is expressed only in that the points of compensation we have selected (i.e., the two positions of the mechanism for which we should obtain complete compensation in accordance with our calculations) will be displaced somewhat in accordance with variations in the magnitude of Δt , although this displacement will be quite insignificant if the given temperature range is not excessive. This is of course of no importance, since the "compensated" points, in general, are given only approximate designations.

In individual special cases one of the given compensations alone is capable of replacing the combined work of both compensations.

Sometimes, for example, it is possible to dispense with primary compensation. ^{STAT}
In actuality, in the majority of cases we note that an instrument not yet compensated for temperature, will show a thermal error of zero at some value of the force

it is measuring. Usually this occurs when the force being measured is zero. Now let this value of the force being measured correspond specifically to a position of the transmitting mechanism at which the work of secondary compensation does not affect the position of the pointer (for a crank in an airspeed indicator, for example, this is the position at which the crank is normal to the connector link). Then, as may be seen from the foregoing, the primary compensation must be zero. Usually, however, this is only a special case.

Likewise, a special case is possible in which primary compensation alone is capable of substituting for the dual compensation. The fact is that we know, from our chapter on general static errors of instruments that the thermal error, expressed in units of the force being measured, is

$$\Delta P = P \beta \Delta t$$

when we are measuring a concentrated force P , and

$$\Delta p = p \beta \Delta t$$

when we are measuring a distributed pressure p .

In order to determine how this error is expressed on variations in deflection Δx of the pickup, we may write in approximate terms

$$\left. \begin{aligned} \Delta x &\approx \Delta P \left(\frac{dx}{dP} \right)_{\text{for a given } P} \\ \text{or} \\ \Delta x &\approx \Delta p \left(\frac{dx}{dp} \right)_{\text{for a given } p} \end{aligned} \right\} \quad (11.8)$$

Now let us take a case in which the following relationship exists between load and travel of the pickup

$$\left. \begin{aligned} x - x_0 &= c (\ln P - \ln P_0) \\ \text{with concentrated load or} \end{aligned} \right\} \quad \text{STAT} \quad (11.9)$$

$$x - x_0 = c_1 (\ln p - \ln p_0) \quad (1)$$

with distributed load.

Here x_0 denotes some specific position of the pickup element, and P_0 or p_0 is the load corresponding to its position, while c is a constant. Then,

$$\frac{dx}{dP} = \frac{c}{P},$$

or

$$\frac{dx}{dp} = \frac{c_1}{p}.$$

Substituting in eq.(11.8) the values obtained for $\frac{dx}{dP}$ or for $\frac{dx}{dp}$, and also the values derived above for ΔP and Δp , we obtain

$$\Delta x \approx P \beta \Delta t \frac{c}{P} = c \beta \Delta t,$$

or

$$\Delta x \approx p \beta \Delta t \frac{c_1}{p} = c_1 \beta \Delta t,$$

i.e., the error Δx does not depend on the quantity to be measured and is constant at all readings of the instrument.

Thus, in this case primary compensation alone may completely compensate the thermal error at all readings of the instrument.

If eq.(11.9) is not fully satisfied at all readings of the instrument, but is satisfied for the two loads P_0 and P_1 , or p_0 and p_1 , or precisely those for which we wish to produce complete compensation of the instrument error, i.e., if we only have

$$x_1 - x_0 = c (\ln P_1 - \ln P_0),$$

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or

$$x_1 - x_0 = c_1 (\ln p_1 - \ln p_0),$$

for these loads, then primary compensation alone is adequate for a complete compensation of the thermal error for two of these loads, while for the others the bulk of the error will be compensated. In other words, we will obtain the same results usually reached with double compensation.

Of course, an analogous situation may be encountered when the error Δp is expressed by the equation

$$\Delta p = -(b - p\beta) \Delta t$$

(see Chapter on the total static instrument errors).

Obviously, the existence of these special cases is demonstrated on combined solution of the two equations of the system (11.7) for the unknowns Δc and Δa at two readings of the instrument (for which we calculate the thermal compensation). This is expressed in the fact that either the unknown Δc or the unknown Δa is equal to zero in solving these equations.

In the mass production of instruments it will often happen that the thermal deflections required of both types of compensation at any given changes in temperature will differ from one specimen to the next. Therefore, provisions for calibration of both types of compensation have to be made.

To adjust the primary compensation, the bimetal strip (2) which provides the compensation (Fig. 85), is free to rotate, with the roller (3) in which the bimetal strip is mounted, about the axis of this roller. As may be seen from the illustration, this spindle is normal to the link (4) and parallel to the plane of the diaphragm (1). A certain amount of effort is required to perform this rotation, so as to render it impossible for the bimetal strip to rotate as the instrument functions.

The hinged joint between the end of the bimetal strip and the link prevents a rotation of the strip from interfering with or impeding the functioning of the hinge. STAT

A rotation of the bimetal strip in this fashion makes it possible to produce

different angles between the link (4) and the direction of the path of the end of the bimetal strip when bent by temperature. It is obvious that, in a case in which the direction of this path coincides with the direction of the link (4), the effect of compensation will be at its greatest. In a case in which the path is normal to the link, the compensation hardly affects the instrument.

In deriving eq.(11.7), we assumed that the direction of the deflection Δc of the bimetal strip coincided with the direction BD. As a result, eq.(11.7) provides only for the projection $\Delta c \cos \delta$ of this deflection in the direction BA of the connector link.

Now let us suppose that the bimetal strip undergoes a deflection Δc_1 in some di-

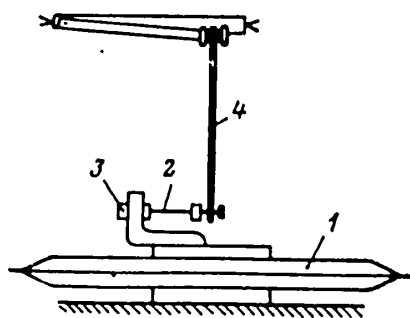


Fig. 85 - Diagram of Kinematic Bimetal Temperature Compensation

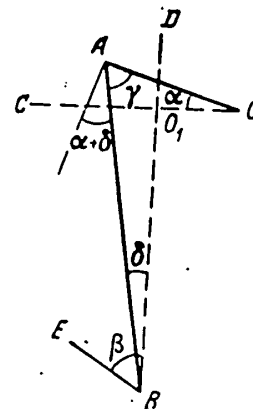


Fig. 86 - See Calculation of Kinematic Bimetal Temperature Compensation

rection BE (Fig. 86), forming an angle δ with BD. In this case, a change in temperature will cause changes not only in the values of $O_1 B$ and a in eq.(11.1), but also of d . However, the values of Δc [see the definitions for eq.(11.5)] and Δd are correlated in the following manner:

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$$\frac{\Delta d}{\Delta c} = \operatorname{tg} \beta;$$

(11.10)

$$\left. \begin{aligned} \Delta c &= \Delta c_1 \cos \beta; \\ \Delta d &= \Delta c_1 \sin \beta. \end{aligned} \right\}$$

In the given instance, eq.(11.6) takes on the form

$$\Delta(O_1B) \approx \frac{\partial(O_1B)}{\partial a} \Delta a + \frac{\partial(O_1B)}{\partial d} \Delta d. \quad (11.11)$$

From this, by employing the above transformations, we obtain

$$\begin{aligned} \Delta x - \Delta c &= -\frac{\cos \gamma}{\cos \delta} \Delta a + \frac{\frac{a \cos \alpha - d}{b}}{\sqrt{1 - \left(\frac{a \cos \alpha - d}{b}\right)^2}} \Delta d = \\ &= -\frac{\cos \gamma}{\cos \delta} \Delta a + \frac{\sin \delta}{\cos \delta} \Delta d. \end{aligned} \quad (11.12)$$

Substituting the equations of the system (11.10) in this equation and multiplying the equations by $\cos \delta$, we obtain

$$\Delta x \cos \delta - \Delta c_1 \cos \beta \cos \delta = -\Delta a \cos \gamma + \Delta c_1 \sin \beta \sin \delta,$$

from which we derive

$$\Delta x \cos \delta \approx \Delta c_1 \cos(\beta - \delta) - \Delta a \cos \gamma. \quad (11.13)$$

The angle $\beta - \delta$ is the angle between the direction ED of the path of the end of the bimetal strip when bent, and the connector link BA . Thus, by varying the magnitude of this angle, we obtain a different value for the effect of bending of the bimetal strip, at a particular instrument reading.

Secondary compensation is also subject to adjustment. The design of this element and regulation thereof have already been described in the Chapter on bimetal strips.

To calculate the kinematic thermal bimetal compensation, the equations derived in this Chapter and the formulas obtained in the Chapter on bimetal strips, must be

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supplemented by expressions for the change in the deflection of pickups with variations in temperature. From the Chapter on instrument errors we already know that, for approximated calculation of this value, we may employ the formulas

$$\left. \begin{aligned} \Delta x &\approx \frac{P\beta\Delta t}{\frac{dP}{dx}(1-\beta\Delta t)} \\ \Delta x &\approx \frac{p\beta\Delta t}{\frac{dp}{dx}(1-\beta\Delta t)} \end{aligned} \right\} \quad (11.14)$$

for concentrated load and

for distributed load, where the value $(1 - \beta\Delta t)$ in the denominators of these equations usually is almost unity, so that it is often possible to neglect this factor in approximated calculations of thermal errors.

The general procedure for calculating the kinematic double bimetal temperature compensation is as follows:

Let us assign two values, M_1 and M_2 , to the value to be measured, at which this double compensation should completely compensate the thermal error of the instrument. If the primary value to be measured is not a force or a pressure (as in an altimeter, for example), then the appropriate equations will easily permit a determination of two forces or two pressures, corresponding to the two given values of the quantity to be measured. Then, from the equations in the system (11.14) it is possible to determine the temperature-induced changes Δx_1 and Δx_2 in the deflection of the pickup corresponding to the values M_1 and M_2 of the quantity being measured.

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In addition, the calculation of the transmitting mechanism, which should be completed prior to calculating the thermal compensation, it is necessary to determine the approximate value of the angles δ and γ , corresponding to the value M_1 of the value being measured (we will call these γ_1 and δ_1), and also to find the value of these angles corresponding to the value M_2 of the quantity being measured (these we will call γ_2 and δ_2).

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Then, for compensation of the thermal errors of the instrument at two given values δ_1 and δ_2 of the value to be measured, two equations must be satisfied:

$$\Delta x_1 \cos \delta_1 = \Delta c_1 \cos (\beta - \delta_1) - \Delta a \cos \gamma_1;$$

$$\Delta x_2 \cos \delta_2 = \Delta c_1 \cos (\beta - \delta_2) - \Delta a \cos \gamma_2.$$

Having assigned some mean angle of adjustment, τ (e.g., 40°), we then proceed to a combined solution of these two equations for the unknowns Δc_1 and Δa . The resultant value of Δc_1 is the thermal deflection we have been seeking for the end of the bimetal strip, providing primary compensation. The value obtained for Δa is the mean magnitude required for thermal deflection of the end of the bimetal strip in secondary compensation. In other words, when the screw (3) (Fig. 65) is mounted in one of the extreme adjustment apertures, the thermal deflection of the end of the given bimetal strip must be larger than the value of Δa obtained from the equations, and when the screw (3) is mounted in the other extreme adjustment aperture this deflection must, on the other hand, be smaller than the value of Δa .

Having determined the required values of Δc_1 and Δa , it is possible, on the basis of the above-derived formulas of bimetal compensation to determine the dimensions of the bimetal strips required.

Section 2. Thermal Bimetal "Force" Compensation of Instruments

Until very recently, the use of bimetal thermal compensation to reduce temperature errors in instruments has usually been effected by introducing bimetal links in the transmitting mechanism of the instrument. In this solution, the function of the bimetal temperature compensation amounted to producing a change in the transmission ratio of the mechanism and to displace its initial position. This type of compensation is known as "kinematic" compensation.

In addition, instrument designs have recently been placed on the market, with bimetallic temperature compensation of a somewhat different principle and arrange-

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ment. In these designs, the effect of temperature compensation consists in having the bimetal spring exercise a specific force on the pickup of the instrument, where the exerted pressure varies with changes in temperature so as to compensate the variation in the modulus of elasticity of the material constituting the spring-like portion of the pickup. This results in a marked reduction of the thermal variations in the deflection of the pickup element. This type of compensation, as distinct from the kinematic compensation referred to above, may be termed a thermal bimetal "force" compensation.

Figures 87 and 88 present the general design of the latest instruments with thermal bimetal "force" compensation (Fig.87 presenting an instrument using an aner-

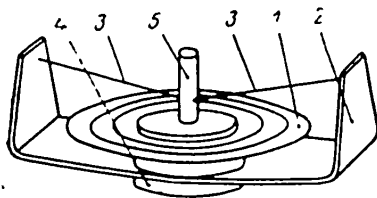


Fig.87 - Diagram of Bimetal Thermal Force Compensation in an Aneroid Cell

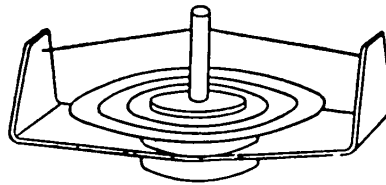


Fig.88 - Diagram of Bimetal Thermal Force Compensation in a Box for Differential Pressure Measurement

oid cell as the pickup element, and Fig.88 one with a pressure-gage type of box). It is clear that both illustrations represent one and the same scheme of thermal compensation in principle, since in both cases the compression needles (3) are mounted in such a fashion that the equi-action compressive forces of all the needles n^2 , increase with a rise in the deflection of the pickup with any increase in STAT the pressure to be measured [here n is the number of needles (Fig.87 (3))]: in the given design, one may conceive of a bimetal spring (2) having more than two ends, arranged symmetrically relative to the axis of the box, and a corresponding number

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of compression needles].

It may also be noted that a completely identical system of bimetal thermal compensation can be applied not only in instruments in which the pickup is in the form of a pressure box, but in general in instruments in which a pickup is employed to measure some force.

Thus, the principles of the compensation shown in Figs. 87 and 88 are the following. A bimetal spring (2) (shown in Figs. 89 and 90 as bimetal yokes) is mounted on the same base as the pickup (1). Resting in this bimetal spring are the ends of the

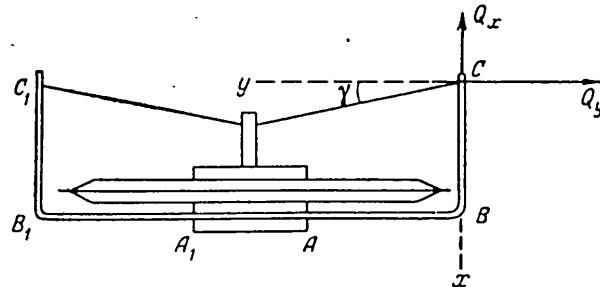


Fig. 89 - See Calculation of Bimetal Thermal Force Compensation

compression needles (3) whose other ends are mounted in the movable portion of the pickup (in Figs. 87 and 88), specifically in the shaft of the rigid center of the pressure box. The ends of the needles remain at specific points in the bimetal spring and the shaft of the rigid center of the box since they have been inserted in corresponding sockets (punches) in these parts. The dimensions of the springs and compression needles have been set so as to produce an initial tension in the bimetal spring (2); this prevents the needles from dropping out. In fact, they even continue to exert pressure on the pickup during thermal flexure of the bimetal spring and on maximum deflection of the pickup. STAT

The following is the operating principle of this device: On any change in tem-

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perature, the force nQ_{xx} (the reduced compressive force exerted by the bimetal spring on the pickup) varies in magnitude as a result of thermal bending of the bimetal spring. In this case, the variation indicated $(\Delta Q_{xx})_t$ in the reduced force Q_{xx} , transmitted to each of the needles (3), will be larger at greater deflections of the pickup due to the effects of the force to be measured, i.e., at greater forces to be measured. This condition (which is necessary for the above type of compensation to function properly) is satisfied, as indicated in Figs. 87 and 88, by so placing the compression needles that the angle γ between the needles and a plane normal to the path of the pickup increases with any increase in the deflection of the pickup during an increase in the force to be measured.

In fact, as the angle γ increases (Fig. 89) there is an increase in the direction x of the deflection of the pickup on the part of the $(\Delta Q_{xx})_t$ component of the thermal variation $(\Delta Q)_{xx}$ in the force of compression Q of the needle, since the force of compression of the needle is directed along the needle, so that $(\Delta Q_{xx})_t = (\Delta Q)_{xx} \sin \gamma$. However, this increase in force $n(\Delta Q_{xx})_t$, along with an increase in the force to be measured, is necessary since the thermal errors of other elastic elements, expressed in units of the force to be measured are, as we know, equal to $(\Delta P)_t = P \alpha \Delta t$ (in the case of pressure measurement $(\Delta p)_t = p \alpha \Delta t$), where α is the thermal coefficient of the modulus of elasticity of the material constituting the elastic pickup, while Δt is the variation in temperature, i.e., $(\Delta P)_t$ also rises with an increase in the force P to be measured.

In this Chapter we are giving a method for calculating a bimetal thermal "force" compensation of the type described. We will assume, in this connection, that the bimetal spring has the shape of a rectangular yoke, as shown in Figs. 87 and 88 (segments 88 and 81 of the spring in Fig. 89 making approximately a right angle with the center section of the spring). Let us note, further, that the bimetal spring in recent instruments with thermal bimetallic force compensation, have had specifically this shape.

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Since the center portion, AA_1 (see Fig.89) of the spring is fixed, we assume in our calculations that two identical springs, ABC and $A_1B_1C_1$ (the free, unfixed portions of the spring), act on the two sides of the pickup and that their angles A_1BC and $A_1B_1C_1$ are approximately equal to a right angle. The springs ABC and $A_1B_1C_1$ are always under some tension, due to the needles (3) resting therein (see Fig.87). The following forces act on each of the springs: Q_x in a direction opposite to the deflection of the pickup (considering this deflection to be positive in the direction of action of the pressure being measured), and a force Q_y in the direction normal thereto, while

$$\frac{Q_x}{Q_y} = \operatorname{tg} \gamma = \frac{z + f_x}{l + f_y},$$

where f_x and f_y are the deflections of the point C of the spring in the direction of the forces Q_x and Q_y (i.e., in the negative direction from the coordinates x and y , if we consider the directions Q_x and Q_y shown in Fig.89 to be positive);

l is a constant (at constant temperature) depending on the initial dimensions of the bimetal spring in the unloaded state:

z is the sum of some initial constant at constant temperature, of a magnitude of ϵ , depending on the initial position of installation of the needles (3) and of the deflection γ of the box: ($z = \epsilon + \gamma$).

When the deflection of the pickup changes, while the temperature remains constant, the values of Q_x , Q_y , f_x and f_y also change, but these four variables are always interrelated by the following equations

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$$\left. \begin{aligned} \frac{Q_x}{Q_y} &= \frac{z + f_x}{l + f_y}; \\ (z + f_x)^2 + (l + f_y)^2 &= L^2; \\ f_x &= D_1 Q_x - D_2 Q_y; \\ f_y &= D_2 Q_y - D_3 Q_x, \end{aligned} \right\} \quad (11.15)$$

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where L is the length of the needle (3).

The values D_1 , D_2 and D_3 are specific magnitudes depending on the parameters of the bimetal spring.

If, for purposes of simplification of the calculation, we disregard the longitudinal flexure of the spring elements, then, in accordance with Costigliano's theorem, we will have for a spring of constant thickness (and taking point C as the origin of coordinates), the following expressions for determining the values D_1 , D_2 and D_3

$$\left. \begin{aligned} D_1 &= k \int \frac{y^2 ds}{b}; \\ D_2 &= k \int \frac{x^2 ds}{b}; \\ D_3 &= k \int \frac{xy ds}{b}, \end{aligned} \right\} \quad (11.16)$$

in which the integrals are determined within limits along the entire length of the line CBA of the center section of the spring in Fig. 89.

Here ds is a segment of this line, of infinitesimal length;

b is the width of the spring, which may vary along the length of the spring

(e.g., in a spring of trapezoidal form, a shape very often used in the design under examination) and is some specific function of the coordinates x or y ;

k is a constant factor for a bimetal spring; regardless of the ratio of the thickness of the two layers of metal comprising the strip its form is

$$k = \frac{12(E_1 h_1 + E_2 h_2)}{4 E_1 E_2 h_1 h_2 (h_1 + h_2)^2 + (E_1 h_1^2 - E_2 h_2^2)^2},$$

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where E_1 is the modulus of elasticity, and h_1 the thickness, of one of the metals,

E_2 is the modulus of elasticity of the other, and h_2 its thickness.

We know that, in the production of bimetal strips, the condition that $\frac{h_1}{h_2} = \frac{\sqrt{E_2}}{\sqrt{E_1}}$ is usually obeyed since, under this condition, the bimetal strip will offer

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maximum thermal flexure (the total thickness of the strip, $h = h_1 + h_2$, being unchanged).

If this condition is satisfied, we have

$$k = \frac{3(E_1 h_1 + E_2 h_2)}{E_1 E_2 h_1 h_2 (h_1 + h_2)^2}$$

Further transformation of this expression, by means of equation $\frac{h_1}{h_2} = \frac{\sqrt{E_2}}{\sqrt{E_1}}$ yields

$$\begin{aligned} k &= \frac{3(E_1 h_1 + E_2 h_2)(h_1 + h_2)}{E_1 E_2 h_1 h_2 (h_1 + h_2)^3} = \frac{3(E_1 \sqrt{E_2} + E_2 \sqrt{E_1})(\sqrt{E_1} + \sqrt{E_2})}{E_1 E_2 \sqrt{E_1} \sqrt{E_2} (h_1 + h_2)^3} = \\ &= \frac{3(\sqrt{E_1} + \sqrt{E_2})^2}{E_1 E_2 h^3}, \end{aligned}$$

where $h = h_1 + h_2$ is the total thickness of the strip.

In view of the fact that, in monometal strips, the value of k corresponds to the expression $\frac{12}{E_b h^3}$, we can write for bimetal strips (let us remember that we adhere here to the condition $\frac{h_1}{h_2} = \frac{\sqrt{E_2}}{\sqrt{E_1}}$, which is, in fact, always approximately satisfied):

$$k = \frac{12}{E_b h^3},$$

where E_b is the reduced modulus of elasticity of the bimetal strip, equal to

$$E_b = \frac{4E_1 E_2}{(\sqrt{E_1} + \sqrt{E_2})^2}. \quad (11.17)$$

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In the case of a trapezoidal bimetal spring, let us assume, in first approximation and for the sake of simplicity, that the segments OB and BA of the spring are linear and that the segment OB agrees in direction with the direction of the axis Ox , while the segment BA is parallel to the axis Oy . Ignoring a certain radius of curvature of the spring at the point B, we obtain

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$$\left. \begin{aligned} D_1 &= k \int_0^c \frac{y^3 dy}{b_1 + m_1 y}; \\ D_2 &= k \left\{ \int_0^a \frac{x^3 dx}{b_2 + m_2 x} + a^2 \int_0^c \frac{dy}{b_1 + m_1 y} \right\}; \\ D_3 &= ka \int_0^c \frac{y dy}{b_1 + m_1 y}, \end{aligned} \right\} \quad (11.18)$$

where a is the length of the spring segment BC;

c is the length of the segment AB;

b_0 , b_1 and b_2 are the values for the width of the spring at the points A, B and C, respectively:

$$\begin{aligned} m_1 &= \frac{b_0 - b_1}{c}; \\ m_2 &= \frac{b_1 - b_2}{a}. \end{aligned}$$

After integration of the equations in the system (11.18), we obtain

$$\begin{aligned} D_1 &\approx \frac{12}{E_b h^3 m_1^2} \int_0^c \frac{m_1^2 y^3 dy}{b_1 + m_1 y} = \frac{12}{E_b h^3 m_1} \left\{ \frac{b_1^2}{m_1^2} \ln \frac{b_0}{b_1} - \frac{b_1}{m_1} c + \frac{c^2}{2} \right\} = \\ &= \frac{12 c^3}{E_b h^3 (b_0 - b_1)} \left[\left(\frac{b_1}{b_0 - b_1} \right)^2 \ln \frac{b_0}{b_1} - \frac{b_1}{b_0 - b_1} + \frac{1}{2} \right]; \\ D_2 &\approx \frac{12}{E_b h^3} \left\{ \int_0^a \frac{x^3 dx}{b_2 + m_2 x} + a^2 \int_0^c \frac{dy}{b_1 + m_1 y} \right\} = \\ &= \frac{12}{E_b h^3} \left\{ \frac{a^3}{b_1 - b_2} \left[\left(\frac{b_2}{b_1 - b_2} \right)^2 \ln \frac{b_1}{b_2} - \frac{b_2}{b_1 - b_2} + \frac{1}{2} \right] + \frac{ca^2}{b_0 - b_1} \ln \frac{b_0}{b_1} \right\}; \\ D_3 &\approx \frac{12a}{E_b h^3 m_1} \int_0^c \frac{m_1 y dy}{b_1 + m_1 y} = \frac{12a}{E_b h^3 m_1} \left[c - \frac{b_1}{m_1} \ln \frac{b_0}{b_1} \right] = \\ &= \frac{12ac^2}{E_b h^3 (b_0 - b_1)} \left[1 - \frac{b_1}{b_0 - b_1} \ln \frac{b_0}{b_1} \right]. \end{aligned} \quad (11.19)$$

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The values D_1 , D_2 and D_3 may also be expressed in the following form:

$$\left. \begin{aligned} D_1 &\approx \frac{12a^3}{E_b b_0 h^3} N_1; \\ D_2 &\approx \frac{12a^3}{E_b b_0 h^3} N_2; \\ D_3 &\approx \frac{12a^3}{E_b b_0 h^3} N_3, \end{aligned} \right\} \quad (11.20)$$

where b_0 is the width of the spring at the point A, where it is attached:

N_1 , N_2 and N_3 are dimensionless magnitudes of the form

$$\left. \begin{aligned} N_1 &= \left(\frac{c}{a}\right)^3 \frac{b_0}{b_0 - b_1} \left[\left(\frac{b_1}{b_0 - b_1}\right)^2 \ln \frac{b_0}{b_1} - \frac{b_1}{b_0 - b_1} + \frac{1}{2} \right]; \\ N_2 &= \frac{b_0}{b_1 - b_2} \left[\left(\frac{b_2}{b_1 - b_2}\right)^2 \ln \frac{b_1}{b_2} - \frac{b_2}{b_1 - b_2} + \frac{1}{2} \right] + \frac{c}{a} \frac{b_0}{b_0 - b_1} \ln \frac{b_0}{b_1}; \\ N_3 &= \left(\frac{c}{a}\right)^2 \frac{b_0}{b_0 - b_1} \left[1 - \frac{b_1}{b_0 - b_1} \ln \frac{b_0}{b_1} \right]. \end{aligned} \right\} \quad (11.20a)$$

We note, further, that in these expressions the value of b_1 is usually equal to $\frac{ab_0 + cb_2}{a + c}$, since the entire spring is usually made of one trapezoidal shape, which is then bent at the section b_1 , i.e., $m_1 = m_2$. Therefore, if we are first to compile Tables for the calculated values of N_1 , N_2 and N_3 , they may be limited to the case in which $b_1 = \frac{ab_0 + cb_2}{a + b}$, and in calculating the Tables, only the various ratios of $\frac{c}{a}$ and $\frac{b_2}{b_0}$ need be assigned.

In the case of a rectangular spring ($b_0 = b_1 = b_2$), the expressions D_1 , D_2 and D_3 are simpler in form, and read

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$$\left. \begin{aligned} D_1 &= \frac{4c^3}{E_b b h^3}; \\ D_2 &= \frac{12}{E_b b h^3} \left[\frac{a^3}{3} + a^2 c \right]; \\ D_3 &= \frac{6ac^2}{E_b b h^3}. \end{aligned} \right\} \quad (11.21)$$

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The case of a rectangular spring is, of course, a special case of the trapezoidal spring. Nevertheless, eqs.(11.19) or (11.20) in the form in which they were written above, are not transformed into eq.(11.21) when $b_0 = b_1 = b_2$, and the terms U_1 , U_2 and U_3 are, in this case, equal to an indeterminate quantity. Nevertheless, these formulas are accurate and should be used in calculating for cases of $b_0 \neq b_2$. In general, without resorting to the general methods of discovering indeterminate values, we may rewrite the eqs.(11.20) in a somewhat more complex form, in which they will undergo conversion to eqs.(11.21) when $b_0 = b_1 = b_2$

$$N_1 = \left(\frac{c}{a}\right)^3 b_0 \left\{ \frac{1}{2(b_0 - b_1)} - \frac{b_1}{(b_0 - b_1)^2} + \frac{b_1^2}{(b_0 - b_1)^3} \left[2 \frac{b_0 - b_1}{b_0 + b_1} + \frac{2}{3} \frac{(b_0 - b_1)^3}{(b_0 + b_1)^3} \right] + \frac{b_1^3}{(b_0 - b_1)^3} \left[\ln \frac{b_0}{b_1} - 2 \frac{b_0 - b_1}{b_0 + b_1} - \frac{2}{3} \frac{(b_0 - b_1)^3}{(b_0 + b_1)^3} \right] \right\} = \left(\frac{c}{a}\right)^3 b_0 \left\{ \frac{3b_0^2 + 6b_0b_1 + 7b_1^2}{6(b_0 + b_1)^3} + \frac{b_1^2}{(b_0 - b_1)^3} \left[\ln \frac{b_0}{b_1} - 2 \frac{b_0 - b_1}{b_0 + b_1} - \frac{2}{3} \frac{(b_0 - b_1)^3}{(b_0 + b_1)^3} \right] \right\}$$

and since

$$\ln \frac{b_0}{b_1} = 2 \left[\frac{b_0 - b_1}{b_0 + b_1} + \frac{1}{3} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^3 + \frac{1}{5} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^5 + \dots \right],$$

therefore

$$N_1 = \left(\frac{c}{a}\right)^3 b_0 \left\{ \frac{3b_0^2 + 6b_0b_1 + 7b_1^2}{6(b_0 + b_1)^3} + \frac{2b_1^2}{(b_0 - b_1)^3} \left[\frac{1}{5} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^5 + \frac{1}{7} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^7 + \dots \right] \right\}.$$

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Analogous transformations yield

$$N_2 = b_0 \left\{ \frac{3b_1^2 + 6b_1b_2 + 7b_2^2}{6(b_1 + b_2)^3} + \frac{b_2^2}{(b_1 - b_2)^3} \left[\ln \frac{b_1}{b_2} - 2 \frac{b_1 - b_2}{b_1 + b_2} - \frac{2}{3} \left(\frac{b_1 - b_2}{b_1 + b_2} \right)^3 \right] \right\} +$$

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$$\begin{aligned}
& + \frac{c}{a} b_0 \left\{ \frac{2}{b_0 + b_1} + \frac{1}{b_0 - b_1} \left[\ln \frac{b_0}{b_1} - 2 \frac{b_0 - b_1}{b_0 + b_1} \right] \right\} = \\
& = b_0 \left\{ \frac{3b_1^2 + 6b_1b_2 + 7b_2^2}{6(b_1 + b_2)^3} + \frac{2b_2^2}{(b_1 - b_2)^2} \left[\frac{1}{5} \left(\frac{b_1 - b_2}{b_1 + b_2} \right)^5 + \frac{1}{7} \left(\frac{b_1 - b_2}{b_1 + b_2} \right)^7 + \dots \right] \right. \\
& \quad \left. + \frac{c}{a} b_0 \left\{ \frac{2}{b_0 + b_1} - \frac{2}{b_0 - b_1} \left[\frac{1}{3} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^3 + \frac{1}{5} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^5 + \dots \right] \right\} \right\}; \\
& N_3 = \left(\frac{c}{a} \right)^2 \frac{b_0}{b_0 - b_1} \left\{ 1 - \frac{2b_1}{b_0 + b_1} - \frac{b_1}{b_0 - b_1} \left[\ln \frac{b_0}{b_1} - 2 \frac{b_0 - b_1}{b_0 + b_1} \right] \right\} = \\
& = \left(\frac{c}{a} \right)^2 b_0 \left\{ \frac{1}{b_0 + b_1} - \frac{b_1}{(b_0 - b_1)^2} \left[\ln \frac{b_0}{b_1} - 2 \frac{b_0 - b_1}{b_0 + b_1} \right] \right\} = \\
& = \left(\frac{c}{a} \right)^2 b_0 \left\{ \frac{1}{b_0 + b_1} - \frac{2b_1}{(b_0 - b_1)^2} \left[\frac{1}{3} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^3 + \frac{1}{5} \left(\frac{b_0 - b_1}{b_0 + b_1} \right)^5 + \dots \right] \right\}.
\end{aligned}$$

When $b_0 = b_1 = b_2$, the second terms within the braces in expressions N_1 , N_2 , and N_3 are reduced to zero, and these expressions are then equal to

$$N_1 = \left(\frac{c}{a} \right)^2 \frac{1}{3};$$

$$N_2 = \frac{1}{3} + \frac{c}{a};$$

$$N_3 = \left(\frac{c}{a} \right)^2 \frac{1}{2},$$

which is in agreement with the equations in the system (11.21).

Returning to an examination of the equations in (11.15), we see that the form of these equations is quite improper for a combined solution for the unknowns b_x , Q_x , f_x , and f_y (to obtain expression Q_x in the form of a function of the quantities D_1 , D_2 , D_3 , l and $z = s + W$, which is required for calculating compensations) STAT

Therefore, let us solve the problem in approximate form (since, in this design, the possibility of adjustment is provided): let us disregard the change $(\Delta f_x)_t$ in the deflection f_x of the bimetal spoke in the directions Ox , produced by the forces Q_x and Q_y due to changes in temperature. This approximation is made on the assumption that, in general, even this entire deflection f_x is quite small relative to the deflection f_y and other quantities with which it is combined in the above equations.

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This is evident from the fact that the force Q_x is considerably smaller than Q_y due to the small size of the angle γ , while, moreover, the deflections in the directions Ox , due to the effects of the forces Q_x and Q_y , compensate each other to a significant degree. The numerical calculations confirm the ideas advanced above.

On this assumption, we obtain, instead of eqs. (11.15), a system of equations with three unknowns Q_x , Q_y , and f_y , in the form

$$\left. \begin{aligned} f_y &= D_1 Q_y - D_2 Q_x; \\ z^2 + (l + f_y)^2 &= L^2; \\ \frac{Q_x}{Q_y} &= \frac{z}{l + f_y} \end{aligned} \right\} \quad (11.22)$$

Note: A more detailed derivation, with a solution of the entire system of equations in (11.15), i.e., one that also allows for the magnitude of $(\Delta f_x)_t$, is made in the author's article on binetal thermal force compensation (Bibl.11).

Eliminating the unknown f_y from these equations, we obtain

$$\begin{aligned} z \frac{Q_y}{Q_x} &= l + D_1 Q_y - D_2 Q_x; \\ z^2 \left[1 + \left(\frac{Q_y}{Q_x} \right)^2 \right] &= L^2, \end{aligned}$$

and therefore

$$\frac{Q_y}{Q_x} = \sqrt{\left(\frac{L}{z} \right)^2 - 1},$$

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from which we derive

$$Q_x = \frac{\sqrt{L^2 - z^2} - l}{D_2 \sqrt{\left(\frac{L}{z} \right)^2 - 1} - D_1} = \frac{E_b b_0 h^3 (\sqrt{L^2 - z^2} - l)}{12 a^3 (N_2 \sqrt{\left(\frac{L}{z} \right)^2 - 1} - N_1)} \quad (11.23)$$

With a change in temperature, the values l and z change due to the thermal de-

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flection of the bimetal strip, and the magnitude of the force Q_x also changes accordingly. Further, temperature fluctuations also cause a slight change in the magnitude of the force Q_x as a result of changes in the moduli of elasticity of the materials of which the bimetal strip is composed. Considering all this, we may write

$$\begin{aligned}
 (dQ_x)_t &= \frac{\partial Q_x}{\partial z} (dz)_t + \frac{\partial Q_x}{\partial l} (dl)_t + \frac{\partial Q_x}{\partial E_1} dE_1 + \frac{\partial Q_x}{\partial E_2} dE_2 = \frac{E_b b_0 h^3}{12a^3} \times \\
 &\times \left\{ -\frac{z}{\sqrt{L^2 - z^2}} \left(N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3 \right) + N_2 \frac{L^2}{z^3 \sqrt{\left(\frac{L}{z}\right)^2 - 1}} (\sqrt{L^2 - z^2} - l) \right. \\
 &\left. - \frac{1}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3} \right\} dz - \\
 &\left\{ \frac{\partial Q_x}{\partial E_1} dE_1 + \frac{\partial Q_x}{\partial E_2} dE_2 \right\} = \\
 &= \frac{E_b b_0 h^3}{12a^3} \left\{ \frac{N_2 \left[-1 + \left(\frac{L}{z}\right)^2 - \frac{l}{z} \left(\frac{L}{z}\right)^2 \right] - N_3 \frac{1}{\sqrt{\left(\frac{L}{z}\right)^2 - 1}}}{\left(N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3 \right)^2} dz - \right. \\
 &\left. - \frac{1}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1}} dl \right\} + \frac{\partial Q_x}{\partial E_1} dE_1 + \frac{\partial Q_x}{\partial E_2} dE_2 = \\
 &= \frac{E_b b_0 h^3}{12a^3} \left(\left(\frac{N_2 \left\{ \left[\left(\frac{L}{z}\right)^2 - 1\right]^{\frac{3}{2}} - \frac{l}{L} \left(\frac{L}{z}\right)^3 \right\} - N_3}{\sqrt{\left(\frac{L}{z}\right)^2 - 1} \left(N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3 \right)^2} dz - \right. \right. \\
 &\left. \left. - \frac{1}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1}} dl \right) \right) + \frac{\partial Q_x}{\partial E_1} dE_1 + \frac{\partial Q_x}{\partial E_2} dE_2. \quad (11.24)
 \end{aligned}$$

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Further,

$$\begin{aligned}\frac{\partial Q_x}{\partial E} &= \frac{Q_x}{E_b} \frac{\partial E_b}{\partial E_1} = Q_x \frac{\frac{E_2}{(\sqrt{E_1} + \sqrt{E_2})^2} - \frac{E_1 E_2}{(\sqrt{E_1} + \sqrt{E_2})^3 \sqrt{E_1}}}{\frac{E_1 E_2}{(\sqrt{E_1} + \sqrt{E_2})^2}} \\ &= Q_x \frac{(\sqrt{E_1} + \sqrt{E_2}) - \sqrt{E_1}}{E_1 (\sqrt{E_1} + \sqrt{E_2})} = Q_x \frac{\sqrt{E_2}}{E_1 (\sqrt{E_1} + \sqrt{E_2})}\end{aligned}$$

Analogously,

$$\frac{\partial Q_x}{\partial E_2} = \frac{\sqrt{E_1}}{(\sqrt{E_1} + \sqrt{E_2}) E_2} Q_x$$

Since, on a change in temperature,

$$\begin{aligned}dE_1 &= -E_1 \beta_1 dt; \\ dE_2 &= -E_2 \beta_2 dt,\end{aligned}$$

where β_1 and β_2 are the temperature coefficients of the moduli of elasticity of the two metals in the bimetal strip (the moduli of elasticity being, respectively, E_1 and E_2), and dt is the change in temperature, we have

$$\frac{\partial Q_x}{\partial E_1} dE_1 + \frac{\partial Q_x}{\partial E_2} dE_2 = -\frac{\sqrt{E_2} \beta_1 + \sqrt{E_1} \beta_2}{\sqrt{E_1} + \sqrt{E_2}} Q_x dt. \quad (11.25)$$

Since, in a monometal spring, the variation in its tensile strength on temperature changes dt is equal to $P \beta dt$, where β is the thermal coefficient of the modulus of elasticity of the spring material and P is the tensile stress of the spring at normal temperature, we can write

$$\frac{\partial Q_x}{\partial E_1} dE_1 + \frac{\partial Q_x}{\partial E_2} dE_2 = \beta_b Q_x dt, \quad (11.26)$$

where β_b is the reduced thermal coefficient of the reduced modulus of elasticity of the bimetal strip, which we consider to be equal to

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$$\left. \begin{aligned} \beta_b &= \frac{\sqrt{E_2} \beta_1 + \sqrt{E_1} \beta_2}{\sqrt{E_1} + \sqrt{E_2}}, \\ \beta_b &= \frac{h_1 \beta_1 + h_2 \beta_2}{h_1 + h_2}. \end{aligned} \right\} \quad (11.27)$$

or in another form

In eq.(11.24), the values dl_t and dz_t are equal to

$$\left. \begin{aligned} dl_t &= A_2 dt; \\ dz_t &= A_1 dt, \end{aligned} \right\} \quad (11.28)$$

where A_1 and A_2 are expressions representing specific functions of the parameters of a bimetal spring.

As we know, such expressions have the following general form:

$$\left. \begin{aligned} A_2 &= \frac{3}{2} \frac{\alpha_1 - \alpha_2}{h} \int x ds; \\ A_1 &= -\frac{3}{2} \frac{\alpha_1 - \alpha_2}{h} \int y ds, \end{aligned} \right\} \quad (11.29)$$

where α_1 is the temperature coefficient of elongation of one of the components of the bimetal strip;

α_2 is the temperature coefficient of elongation of the other component. Here

the integrals are also obtained within the entire length CB of the middle

layer of the spring in Fig.89 as the limits;

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ds is a section of that line of infinitesimal length.

Assuming here, too, for the sake of simplicity, that the segments CB and BA of the spring are straight and also that the segment CB coincides with the direction of the axis Ox and that the segment BA is parallel to the axis Oy , and in addition, disregarding a certain radius of curvature in the spring at the point B , we finally have

$$A_2 = \frac{3}{2} \frac{\alpha_1 - \alpha_2}{h} \left[\int_0^a x dx + a \int_0^c dy \right] = \frac{3}{2} \frac{\alpha_1 - \alpha_2}{h} \left[\frac{a^2}{2} + ac \right];$$

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$$A_1 = -\frac{3}{2} \frac{a_1 - a_2}{h} \int_0^c y dy = -\frac{3}{2} \frac{a_1 - a_2}{h} \frac{c^2}{2}.$$

The values A_1 and A_2 may also be expressed in the following form:

$$\left. \begin{aligned} A_1 &= -\frac{3}{2} \frac{(a_1 - a_2)}{h} a^2 n_1; \\ A_2 &= \frac{3}{2} \frac{(a_1 - a_2)}{h} a^2 n_2, \end{aligned} \right\} \quad (11.30)$$

where

$$\begin{aligned} n_1 &= \frac{1}{2} \left(\frac{c}{a} \right)^2; \\ n_2 &= \left(\frac{c}{a} + \frac{1}{2} \right). \end{aligned}$$

Substituting expressions (11.26), (11.23), (11.28), and (11.30) in eq.(11.2'), we obtain

$$(dQ_x)_t = -\frac{E_b b_0 h^2}{8a} \left\{ (a_1 - a_2) R + \frac{2}{3} \frac{hL}{a^2} \beta_6 M \right\} dt, \quad (11.31)$$

where

$$\left. \begin{aligned} R &= \frac{n_2}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3} \\ &\quad - \frac{n_1 N_2 \left\{ \left[\left(\frac{L}{z}\right)^2 - 1 \right]^{\frac{3}{2}} - \frac{L}{z} \left(\frac{L}{z}\right)^3 \right\} - n_1 N_3}{\sqrt{\left(\frac{L}{z}\right)^2 - 1} \left(N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3 \right)^2}; \\ M &= \frac{\sqrt{1 - \left(\frac{z}{L}\right)^2} - \frac{L}{z}}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1} - N_3} \end{aligned} \right\} \quad \begin{array}{l} \text{STAT} \\ (11.32a) \end{array}$$

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Further, in accordance with the above assumption,

$$\frac{z}{L} \approx \sin \gamma.$$

Therefore the expressions R and M may also be presented in the form:

$$\left. \begin{aligned} R &= \frac{n_2}{N_2 \operatorname{ctg} \gamma - N_3} - \frac{n_1 N_2 \left(\operatorname{ctg}^2 \gamma - \frac{l}{L \sin^2 \gamma \cos \gamma} \right) - n_1 N_3 \operatorname{tg} \gamma}{(N_2 \operatorname{ctg} \gamma - N_3)^2}; \\ M &= \frac{\cos \gamma - \frac{l}{L}}{N_2 \operatorname{ctg} \gamma - N_3}. \end{aligned} \right\} \quad (11.32)$$

Here R and M are dimensionless values constituting functions of the angle γ and of the ratios $\frac{b_2}{b_0}$, $\frac{l}{L}$ and $\frac{z}{a}$.

For an approximated calculation of compensation, we take the finite difference ΔQ_x and Δt for the differentials (dQ_x) and dt , and obtain the following formulas for the calculation:

$$\Delta Q_x \approx -\frac{E_b b_0 h^2}{8a} \left\{ (\alpha_1 - \alpha_2) R + \frac{2}{3} \frac{hL}{a^2} \beta_b M \right\} \Delta t; \quad (11.33)$$

$$z \approx L \sin \gamma.$$

In order for the force $2(\Delta Q_x)_t$ to compensate the thermal error due to thermal variations in the modulus of elasticity of the substance of which the elastic pickup is made, the following equation must be satisfied:

$$\left(p + 2 \frac{Q_x}{S_{\text{eff}}} \right) \beta_k \Delta t = -2 \frac{(\Delta Q_x)_t}{S_{\text{eff}}}; \quad (11.34) \quad \text{STAT}$$

where S_{eff} is the effective area of the box at a pressure p ,

β_k is the thermal coefficient of the modulus of elasticity of the material of the elastic moment (box).

The requirement thus is that

$$2 \left\{ \frac{E_b b_0 h^2 (\alpha_1 - \alpha_2)}{8a^2 S_{\text{eff}}} R + \frac{E_b b_0 h^3}{12a^3 S_{\text{eff}}} L \frac{\beta_b}{\beta_k} M \right\} = p + 2 \frac{Q_x}{S_{\text{eff}}}, \quad (11.35)$$

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or in other words

$$2 \left\{ \frac{E_b b_0 h^2 (a_1 - a_2)}{8 a \beta_k S_{eff}} R + \frac{Q_x \beta_b}{S_{eff} \beta_k} \right\} = p_0,$$

where $p_0 = p + 2 \frac{Q_x}{S_{eff}}$ is the reduced total pressure on the box.

The best possible temperature compensation at various readings of the instrument within its operating range is reached by finding an adequately satisfactory relationship between the values of $(\Delta \gamma_x)_t$ and z , corresponding to the given characteristic of the pickup, i.e., to the dependence existing between the pressure p_0 and the deflection of the box.

As indicated by specimen calculations and by practical experience, the nature of the chosen ratio between the magnitudes of $(\Delta \gamma_x)_t$ and z is relatively little dependent on the magnitude of the $\frac{c}{a}$ and $\frac{b_2}{b_0}$ ratio [changes in these relationships, for the most part cause some variation only in the total rigidity of a bimetal spring and thus in the total value of $(\Delta \gamma_x)_t$, which is readily compensated by the absence of variations in the width b_0 and the thickness h of the spring]. Therefore, these relationships are usually merely assigned, since a selection of these to obtain even better relationships between the values of $(\Delta \gamma_x)_t$ and z gives no tangible results and would further substantially complicate the matter under examination.

Likewise, specimen calculations and practice demonstrate that any variations in the $\frac{l}{L}$ ratios have practically no influence on the nature of the relationship between the values $(\Delta \gamma_x)_t$ and z (variation in the $\frac{l}{L}$ ratio may have a significant effect on the total value of γ_x but a very limited effect on the value of the variation $(\Delta \gamma_x)_t$ of this pressure with temperature).

Therefore these ratios, too, are usually merely assigned. Moreover, an effort is made to assign, if possible, only such a difference between the quantities L and l that, as indicated above, the compression needles (3) do not drop out and continue to exert pressure on the pickup when the bimetal spring is bent by tempera-

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ture, and when the pickup has undergone extreme deflection. An effort is usually made to make the difference between L and l small, so as not to load the box excessively.

In addition, in determining the parameters of the device under examination, it is possible, for considerations of design and dimensioning, merely to assign the value of L and the mean value of a , which may then be adjusted in accordance with fluctuations in the characteristics of the pickups.

Here we note, further, that the second terms of both parts of eq.(11.35) are only a fraction of the first terms since the pressure $2 \frac{\gamma_{\text{eff}}}{\beta_{\text{eff}}}$ is only a fraction of the measured pressure p , and the variation in the pressure p , due to thermal variations in the modulus of elasticity of a bimetal spring, is only a fraction of the pressure due to the thermal flexure of this spring.

Thus, to arrive at a relationship of the required nature between the values of $(\Delta \gamma_x)_t$ and z , we are fundamentally able to operate only by selecting the two following parameters: b_0^{+2} and the initial fixed angle γ of the compression needles (3). Therefore, as in kinematic double bimetal temperature compensations, we are able here, in the general case, to reach complete temperature compensation only for the two readings of the instrument that are usually assigned in advance for one reason or another. These two readings are known as the compensation points. At other indications, only the basic temperature error (but not that error as a whole) is compensated.

For a calculation of the type of temperature compensation under consideration, ^{STAT} all that is basically necessary is to arrive at the required mean value of b_0^{+2} (as the mean characteristic of a box of the given type), since the other of the above parameters affecting the nature of the relationship between the values $(\Delta \gamma_x)_t$ and z (to wit, the initial angle γ) is selected in the process of adjusting the instrument.

Note: It is quite obvious that, in the design under examination, the provision for adjustment of the initial angle must be supplemented by a provi-

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sion for regulation of some parameter, whose change is equivalent to a variation in the calculated parameter $b_0 h^2$, referred to above. This is necessary in view of the fact that the characteristics of boxes manufactured serially often differ significantly. For this purpose, designs provide for the possibility of varying the total stiffness of the bimetal spring, by adjusting its length.

To calculate thermal compensation it is necessary at the outset to assign the two compensation points. Of course, it would be more correct to attempt determining them by, e.g., the criterion proposed by Zhetyshev. However, this results in a very complex and, most probably, mathematically insoluble undertaking. Therefore, setting a limit by specifying the compensation points at the outset by methods of approximation is necessary.

Thus, the compensation pressures p_1 and p_2 are arrived at, at the outset, on the basis of various considerations.

The compensation pressures p_1 and p_2 , once assigned, should be increased by three or four percent, whereupon these already somewhat elevated pressures may be adopted for use as the total compensation pressures p_{01} and p_{02} , acting on the box (since $p_0 = p + 2 \frac{H}{\sigma_{eff}}$). This procedure is recommended since the calculation has to be conducted in accordance with eq. (11.35), based on the total pressure p_0 .

Note: If, in such an approximated increase in the pressures of p_1 and p_2 by 3 or 4%, we obtain total compensation pressures of p_{01} and p_{02} , not completely corresponding to the compensation pressures p_1 and p_2 , this is of no special significance since the very pressures p_1 and p_2 , as we have just indicated, are approximations. STAT

On the basis of the given characteristic of the pickup, we are in a position to determine its deflections δ_1 and δ_2 , corresponding to these total compensation pressures p_{01} and p_{02} .

After this, from the theoretical point of view, the calculation is reduced to

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satisfaction of the following equations:

$$\left. \begin{aligned} \frac{E_b b_0 h^3 (a_1 - a_2)}{4 a \beta_k S_{eff}} \left[R + \frac{h L \beta_b}{a^2 (a_1 - a_2)} \frac{2}{3} M \right]_{\gamma = \gamma_1} &= p_{01}; \\ \frac{E_b b_0 h^3 (a_1 - a_2)}{4 a \beta_k S_{eff}} \left[R + \frac{h L \beta_b}{a^2 (a_1 - a_2)} \frac{2}{3} M \right]_{\gamma = \gamma_2} &= p_{02}; \\ z_1 - z_2 &= L (\sin \gamma_2 - \sin \gamma_1) \approx W_2 - W_1 \end{aligned} \right\} \quad (11.36)$$

(which neglect, for purposes of approximation, the very insignificant discrepancy in the values $z_1 - z_2$ and $W_2 - W_1$ due to the fact that the deflections f_z of the rakes are not entirely identical at various values of z).

In these three equations, the unknowns are the angles γ_1 and γ_2 and one of the parameters b_0 or h of the bimetal spring. The other parameter may be assigned, on the condition, of course, that if, after solution for the unknowns in the equations of the system (11.36), the parameter b_0 or h proves to be an inapplicable value, the magnitude of the parameter is changed and the calculation is repeated. However, this latter inconvenience occurs primarily because of the fact that the factor h is present in the second term in square brackets of the first equation in the system (11.36). Therefore, in consideration of the fact that this second term in brackets, as noted before, is only a small fraction of the first (usually it is 3 or 4% of the first), it is permissible, for purposes of a first approximation, to omit this second term in calculating compensations.

With this simplification, the unknowns in the three equations of the system (11.36) will be the angles γ_1 and γ_2 and the value $b_0 h^2$. This simplification is further justified by the fact that, in the given design, as indicated, the possibility exists for further compensation of this inaccuracy by adjustment of the total stiffness of the spring, through such measures as changing its length. In addition, after an approximated determination of the quantity $b_0 h^2$ and the choice of the values b_0 and h^2 in accordance with this first approximation, the solution may be rendered more exact by a second approximation, in which the quantity h , calculated by

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the first approximation, is inserted in the second term within the bracket of equations (11.36), and by then again solving that equation.

In this connection, the second approximation may be reduced to very simple manipulations, if the fact is considered that the values of the angles γ_1 and γ_2 undergo very negligible changes in the process of solving for the second approximation and that this insignificant variation in the angles γ_1 and γ_2 should have a particularly insignificant effect in determination of the value of $b_0 h^2$.

In view of these considerations it is possible, in second approximation (which is taken to be quite satisfactory, so no third approximation is suggested) not only to substitute in the bracket of the first or second equation in (11.36) the value of h calculated in accordance with the first approximation, but also to leave untouched the values of γ_1 and γ_2 obtained by the first approximation, with the result that the entire second approximation is reduced to dividing $b_0 h^2$, whose value was determined in first approximation, by the expression

$$1 + \frac{hL\beta_b}{a^2(\alpha_1 - \alpha_2)} \frac{2}{3} \frac{M}{R},$$

i.e.,

$$(b_0 h^2)_{II} \approx (b_0 h^2)_I \frac{1}{1 + \frac{hL\beta_b}{a^2(\alpha_1 - \alpha_2)} \frac{2}{3} \frac{M}{R}}, \quad (11.37)$$

where it makes no difference whether we use $\gamma = \gamma_1$ or $\gamma = \gamma_2$ in this equation. STAT

However, an analytical solution of the equations in (11.36) even to a first equation, i.e., a combined solution of the equations

$$\left. \begin{aligned} \frac{E_b b_0 h^2 (\alpha_1 - \alpha_2)}{4a\beta_k S_{eff}} R_{1=\gamma_1} &= p_{01}; \\ \frac{E_b b_0 h^2 (\alpha_1 - \alpha_2)}{4a\beta_k S_{eff}} R_{1=\gamma_2} &= p_{02}; \\ L(\sin \gamma_2 - \sin \gamma_1) &= W_2 = W_1 \end{aligned} \right\} \quad (11.38)$$

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is also quite difficult, in view of the fact that R is a complex function of the angle γ . Therefore, in the given case the simplest procedure is to resolve the problem graphically. This requires plotting a curve for the ratio of the values R to $\sin \gamma$ so that, with assignment of the different values of γ and calculation of the corresponding magnitudes of R , points with the coordinates R and $\sin \gamma$ are plotted on a graph in which the R values are inscribed on the abscissa and the $\sin \gamma$ values on the ordinate. All these points may then be interconnected with the aid of a French curve, yielding a curve for the relationship between R and $\sin \gamma$.

For this type of calculation of the R coordinates of various points on the curve, it is necessary, once the specific $\frac{c}{a}$ and $\frac{b_2}{b_0}$ ratios have been assigned, to calculate first the values of M_1 , M_2 , and M_3 , and then, while assigning the $\frac{l}{L}$ ratio, to calculate the values of R corresponding to various values of the angle γ . In view of certain difficulties involved in such calculations, offices concerned with the design of instruments of this type must be in possession of prepared plots of the relationship between R and $\sin \gamma$ at the most common $\frac{c}{a}$ ratios (i.e., when $c = a$), for the most commonly used $\frac{b_2}{b_0}$ ratios and for two or three $\frac{l}{L}$ ratios (i.e. for $\frac{l}{L} = 0.9$ and for $\frac{l}{L} = 0.85$). For example, if only one, the most commonly used ratio of $\frac{c}{a}$ and only the two most commonly used $\frac{l}{L}$ ratios are given, all the curves representing the relationship between the values R and $\sin \gamma$ can be plotted on two graphs having approximately the position shown in Fig. 90.

After plotting the graph for the relationship between $\frac{z_1}{L}$ and R , the graphical determination of the compensation parameters may be pursued further in the following manner: Let us plot four lines (Fig. 91) on tracing paper so that the three lines, C_0D_0 , C_5D_5 , and C_6D_6 will be parallel to each other and normal to a fourth line C_0C_6 , and so that the segment C_5C_6 will be equal to the difference between $\frac{W_2}{L} - \frac{W_1}{L}$ in the scale in which the values $\frac{z}{L}$ are plotted in the graph in Fig. 90 for the relationship between $\frac{z}{L}$ and R , and the segment $C_0C_5 = C_5C_6 \frac{Po_1}{Po_2 - Po_1}$ (or in other

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words, the condition that $\frac{C_0C_6}{C_0C_5} = \frac{P_{02}}{P_{01}}$ is adhered to).

The tracing sheet with the system of straight lines for C_0C_6 , C_0D_0 , C_5D_5 , and C_6D_6 is then applied to the graph in Fig.90 for the ratio of $\frac{z}{L}$ to R in such a fashion that the line C_0C_6 coincides with the ordinate $\frac{z}{L}$. Then let us shift the

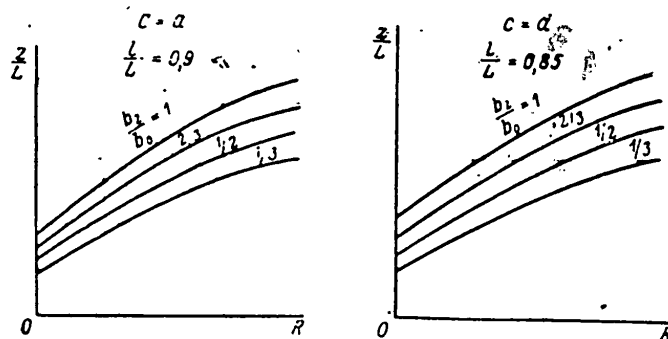


Fig.90 -- Specimen Graphs of the Relationship Between R and the $\frac{z}{L}$ Ratio
(Used in Calculating the Temperature Compensation by Force)

tracing sheet on the graph so that the straight line C_0C_6 moves along the axis $\frac{z}{L}$ (i.e., continues to coincide with the other at all points), and let us at the same time apply a straightedge to the tracing sheet in such a fashion that its edge passes through C_0 and the point of intersection of the line C_5D_5 of the tracing sheet with the curve for the ratio of $\frac{z}{L}$ to R in the graph in Fig.90. The tracing paper should be moved until the point of intersection of the straight line C_6D_6 on the paper and the curve on the graph also appear along the edge of the ruler. In STAT this position (Fig.92), the tracing sheet should be fixed to the graph, and the compensation points C_3 and C_4 should be marked on the graph. With this arrangement, we obviously get the result that the ratio of the segments $\frac{C_6C_4}{C_5C_3}$ in Fig.92 is equal to the ratio $\frac{P_{02}}{P_{01}}$, or that

$$\frac{R_{z=OC_3}}{R_{z=OC_4}} = \frac{P_{02}}{P_{01}}$$

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and in addition, the condition that

$$z_2 - z_1 \approx W_2 - W_1.$$

is satisfied.

Adherence to both these conditions is essential for satisfaction of eqs.(11.38). Having determined the position of the points C_3 and C_4 in Fig.92, it is easy to cal-

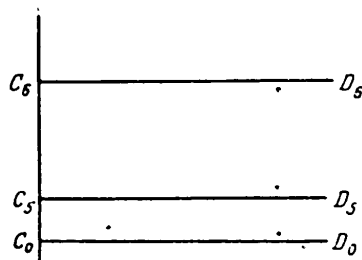


Fig.91 - Plot for Calculating Force-Type Bimetal Temperature Compensations

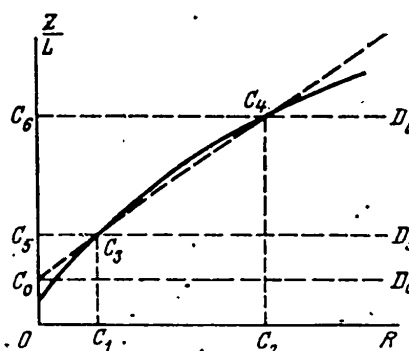


Fig.92 - Plot for Calculating Force-Type Bimetal Temperature Compensations

culate the required value of $b_0 h^2$. In fact, we have, for this purpose, the equation

$$\left. \begin{aligned} \frac{E_b b_0 h^2 (a_1 - a_2)}{4\alpha \beta_K S_{eff}} &= \frac{p_{01}}{R_{z=z_1}} = \frac{p_{01}}{(C_5 C_6) m_R} \\ \text{or, what amounts to the same thing,} \\ \frac{E_b b_0 h^2 (a_1 - a_2)}{4\alpha \beta_K S_{eff}} &= \frac{p_{02}}{R_{z=z_2}} = \frac{p_{02}}{(C_5 C_4) m_R} \end{aligned} \right\} \quad (11.39)_{\text{STAT}}$$

where m_R is the scale on which R is entered into the graphs in Figs.90 and 92.

Having determined the value of $b_0 h^2$, and having selected, on the basis thereof, typical values for b_0 and h , it becomes possible to render the required value of $b_0 h^2$ somewhat more precise by a second approximation, represented by eq.(11.37).

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In addition, the difference between the ordinates of the point C_5 in Fig. 92 and the deflection W_1 is determined by the theoretical calculated initial value z_0 or, more accurately, the adjustable value ϵ (which, in practice, is established experimentally as the instrument is adjusted).

Adjustment of the γ angle of the initial position of the stay pins, and adjustment of the total stiffness of the bimetal spring can be done by aligning several center dots both at the ends of the bimetal spring and on the shaft of the rigid center of the box. This permits a variation both of the total operating length of the spring and of the angle γ .

Recently, a certain variation in the described design has been proposed, consisting in the fact that the segment AB is omitted from the bimetal spring (Fig. 87, 88, and 89), and is replaced by a fairly stiff nonbimetal base underneath the box, to which several bimetal straight springs (2, 3, or even more) are mounted, in which only the segment BC is present. In view of the fact that the deflections of these springs in the direction Cx , due to the effect of the force and due to a change in temperature, can be disregarded, the entire calculation of compensation is considerably facilitated.

In this case, the values n_1 , n_3 , and n_1 may be considered equal to zero, and, in addition (since $C = 0$),

$$N_2 = \frac{b_1}{b_1 - b_2} \left[\left(\frac{b_2}{b_1 - b_2} \right)^2 \ln \frac{b_1}{b_2} - \frac{b_2}{b_1 - b_2} + \frac{1}{2} \right]$$

or (if required)

$$N_2 \approx b_1 \left\{ \frac{3b_1^2 + 6b_1b_2 + 7b_2^2}{6(b_1 + b_2)^3} + \frac{b_2^2}{(b_1 - b_2)^3} \left[\ln \frac{b_1}{b_2} - \frac{8}{3} \frac{(b_1^3 - b_2^3)}{(b_1 + b_2)^3} \right] \right\}$$

and

$$n_2 = \frac{1}{2}$$

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Therefore, we are able to obtain the following expression directly from the equations in the system (11.15), i.e., without substituting the approximated equations in (11.22), and instead of the equations in (11.33):

$$\left. \begin{aligned} (\Delta Q_x)_t &= - \left\{ \frac{E_b b_1 h^2 (a_1 - a_2)}{8a} \frac{1}{2N_2} \operatorname{tg} \gamma + \right. \\ &\quad \left. + \frac{E_b b_1 h^2 (L \cos \gamma - b) \beta_b \operatorname{tg} \gamma}{12a^3 N_2} \right\} \Delta t = \\ &= - \frac{E_b b_1 h^2 (a_1 - a_2)}{16N_2 a} \left[1 + \frac{4h (L \cos \gamma - l) \beta_b}{3a^2 (a_1 - a_2)} \right] \operatorname{tg} \gamma \Delta t; \\ z &= L \sin \gamma. \end{aligned} \right\} \quad (11.40)$$

While, in this special case, we obtain considerably simpler expressions for $(\Delta Q_x)_t$ and z , from which it is not difficult to obtain the expression for the direct relationship between $(\Delta Q_x)_t$ and z , excluding the intermediate parameter γ from these equations, the equations here too are of a structure that is inconvenient for a combined solution. Therefore, here too it is logical to use the same graphic method of solution employed for the general case. Here, however, there is no need to have different curves for the relationship between $\frac{z}{L}$ and R at different $\frac{c}{a}$, $\frac{b_2}{b_0}$, and $\frac{l}{L}$ ratios, and it is possible always to make use of a single curve, which would, furthermore, be very easy to plot. This is the curve describing the difference between the tangent and the sine of the angle, the tangents being plotted along the abscissa at whatever scale is desired, and the sines along the ordinate. Such a curve is very easy to plot, since highly detailed Tables of sines and tangents of angles are generally available.

In this case, if in solving the problem in first approximation, we use the STAT method with four straight lines on tracing paper that we employed previously, and obtain the compensation points C_3 and C_4 on the curve for the relationship between $\sin \gamma$ and $\tan \gamma$, we will then, for determining the value of $b_0 h^2$, have an equation analogous to eq. (11.39)

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$$\left. \begin{aligned} \frac{nE_b b_1 h^2 (a_1 - a_2)}{16 N_2 a \beta_k S_{eff}} &= \frac{p_{o1}}{\gamma_g \gamma_1} = \frac{p_{o1}}{(C_5 C_8) m_{1g}} \\ \text{or} \quad \frac{nE_b b_1 h^2 (a_1 - a_2)}{16 N_2 a \beta_k S_{eff}} &= \frac{p_{o2}}{(C_6 C_4) m_{1g}} \end{aligned} \right\} \quad (11.41)$$

where n is the number of bimetal springs (equal to the number of stay pins),

m_{tan} is the scale on which $\tan \gamma$ has been laid out in the plot.

Section 3. Thermal Compensation by Means of Gas Within the Aneroid Cell

In some altimeters, no hard vacuum is established in the aneroid cells, but a certain amount of gas is permitted to remain, for purposes of compensating temperature errors. In such cases, we note the following:

Let us assume that the temperature has increased. In this case the elasticity of the cell will decrease and, if the differential pressure acting thereon has remained the same as before, the deflection of the box would necessarily increase. However, the gas in the cell would increase in pressure as a result of the elevation of temperature, and therefore in a case in which the external pressure does not increase, the differential pressure acting on the cell decreases. This circumstance compensates, to some degree, for the decline in the elasticity of the box. An analogous picture is created when the cell is cooled.

Here it should be noted that this method of temperature compensation is used rather rarely, since it presents certain manufacturing difficulties (for example, it is very difficult to regulate). However, it is nonetheless used in such instruments as certain recording altimeters, in which bimetal kinematic temperature compensation is undesirable in view of the fact that the reduced force of friction between pen and paper may induce an unnecessary supplementary deflection of the bimetal strip. Therefore, it is necessary to examine this question more closely.

Usually, this compensation is able, by itself, to compensate the temperature

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error of the instrument at only a single reading, and the general effect of this compensation is very similar to that of primary bimetal compensation.

For diaphragm instruments, the thermal error Δp , expressed in units of pressure, is

$$\Delta p \approx p\beta\Delta t,$$

where β is the temperature coefficient of the modulus of elasticity of the material of which the box is composed (see Chapter on total instrument errors).

Thus, with the method of compensation under discussion, it is essential that the differential pressure acting on the cell is specifically reduced by the indicated value of $p\beta\Delta t$. This means that the pressure within the cell must increase by this value.

If the deflection of the cell does not vary, it may be assumed, in first approximation, that its volume also does not vary. In this case, a temperature change will cause the pressure in the cell to vary in proportion to the variations in the absolute temperature of the gas therein, i.e.,

$$\frac{q + \Delta q}{q} = \frac{T + \Delta T}{T},$$

where q is the pressure in the box;

T is the absolute temperature of the contained gas.

From this, we derive

$$\frac{\Delta q}{q} = \frac{\Delta T}{T},$$

or

$$\Delta q = q \frac{\Delta T}{T}.$$

Equating this value to the pressure Δp , we obtain

$$q = T p \beta.$$

(11.12)

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This is the pressure that the gas remaining in the box must have.

The right side of eq.(11.42) includes the differential pressure acting on the box. Thus, different pressures p require different gas pressures in the cell q . It is true that, on variation in the pressure p , aneroid cells change in volume, and, specifically, in the direction necessary for the phenomenon under analysis. It is easy to demonstrate this. However, it should be noted that the relative variation in volume in the majority of aneroid cells, due to deflection, is insufficient to compensate for the effect of the pressure p on the residual pressure q , to be retained in the box.

In view of the foregoing, the presence of a certain amount of gas in the box usually compensates temperature errors only for a single pressure, so that this method may, in actuality, only substitute for primary bimetal compensation.

However, a special case is possible in which the retention of gas in the cell compensates for thermal errors at two pressures, while at other pressures the larger part of this error is compensated, i.e., in general it would replace both types of bimetal compensation. Let us analyze this case.

As indicated before, the compensation must satisfy the equation

$$\Delta q = p \beta \Delta t. \quad (11.43)$$

By weight, the amount of gas Q in the box comes to

$$Q = V \frac{q}{RT},$$

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where V is the volume of the cell,

T is the absolute temperature of the gas therein,

R is the gas constant.

From this, we obtain

$$Vq = QRT.$$

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If the deflection does not vary with changes in temperature (i.e., if temperature compensation has been attained), it may be considered with sufficient accuracy that the volume V also does not change. Then, with any change in temperature, we will have the equation

$$V\Delta q = QR\Delta T = QR\Delta t. \quad (11.44)$$

Comparing eqs. (11.43) and (11.44), we arrive at

$$\frac{QR}{V} = p\beta$$

or

$$Vp = \frac{QR}{\beta}. \quad (11.45)$$

The right side of eq. (11.45) is a constant when the cell is filled with gas to a specific degree. Therefore, if one succeeded in making a chamber which, within a given range of pressures, would undergo changes in volume upon variations in the differential pressure p , so that the product Vp would be approximately constant, such a box could be compensated over the entire range of pressures in question by means of the gas remaining in the box.

Obviously, it would be difficult to create such a cell. But if one could succeed (perhaps by installing appropriate liners), so that the equation

$$V_{p_1}p_1 = V_{p_2}p_2,$$

would be satisfied for two pressures p_1 and p_2 , the cell could be compensated for STAT these two pressures p_1 and p_2 . At all other pressures, the compensation would be partial (and this is what we have when dual bimetal temperature compensation is employed).

But this too is difficult to obtain in practice.

In all the above equations, it is easy to replace the differential pressure p

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by the absolute external pressure p_a , since we are interested in compensation for certain specific external pressures p_a and not for given differential pressures p .

From eq.(11.42) we have

$$p = p_a - q = \frac{q}{\beta T},$$

from which we derive

$$q = \frac{p_a}{1 + \frac{1}{\beta T}};$$

$$p = p_a - q = p_a \left(1 - \frac{1}{1 + \frac{1}{\beta T}} \right) = p_a \frac{1}{1 + \beta T}. \quad (11.46)$$

Substituting this expression in eq.(11.45), we have

$$V p_a = \frac{QR}{\beta} (1 + \beta T). \quad (11.47)$$

In this equation, the value of T is to be considered a constant. This quantity is the temperature at which the cell is filled with gas to a pressure q , in accordance with eq.(11.42). Thus, the right side of eq.(11.47) is constant, i.e., in order for temperature compensation to take place, the following condition must be satisfied at all pressures p_a :

$$V p_a = \text{const.}$$

Substituting the term p from eq.(11.46) in eq.(11.42), we obtain

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$$q = \frac{p_a \beta T}{1 + \beta T}; \quad (11.48)$$

In general outline, we may note the following approach to the construction of such a box, to be compensated for two given pressures p_{a1} and p_{a2} . First let us calculate the differential pressures p_1 and p_2 , related to the pressures p_{a1} and p_{a2}

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by the following equations

$$p_1 = \frac{p_{a1}}{1 + \beta T}$$

and

$$p_2 = \frac{p_{a2}}{1 + \beta T}$$

where T is the temperature at which the cell is filled.

Then, let us determine, in accordance with the cell characteristic, the deflections x_1 and x_2 , corresponding to the differential pressures p_1 and p_2 . Further, if V_0 is the initial volume of the box, we may write

$$V_{p1} = V_0 - f(x_1)$$

and

$$V_{p2} = V_0 - f(x_2),$$

where $f(x_1)$ and $f(x_2)$ are variations in the volume of the box for deflections x_1 and x_2 .

To obtain the temperature compensation of the selected points, it is necessary

$$V_{p1} p_1 = V_{p2} p_2$$

$$[V_0 - f(x_1)] p_1 = [V_0 - f(x_2)] p_2,$$

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from which it follows that

$$V_0 = \frac{f(x_1) p_1 - f(x_2) p_2}{p_1 - p_2} \quad (11.10)$$

or, in another form

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$$V_{p_1} = V_0 - f(x_1) = \frac{[f(x_1) - f(x_2)] p_2}{p_1 - p_2} \quad (11.50)$$

Thus, a solution of our problem requires the initial cell volume (obtained by appropriate design of liners) to be as indicated in eq.(11.49), or as at a pressure of p_1 indicated in eq.(11.50).

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CHAPTER XII

DASHPOTS

In some instruments, special devices are provided to damp the vibrations of the mechanism.

In certain of the simplest instruments (for example, in magnetic compasses and certain instruments of the pendulum type), the vibrations of a magnetic needle or pendulum are attenuated by filling the body of the instrument with liquid:

However, this type of damping involves more rigid sealing of the housing and requires special supplementary devices to compensate thermal expansion of the liquid.

In addition, it is very difficult and sometimes impossible to use this type of damping in the majority of instruments of more complex design. This being the case, a more common solution of the problem of damping the vibrations of a mechanism is the provision of special dampers.

Three types of devices have proved to be the most popular for use in instrument mechanisms:

1. liquid dashpots;
2. pneumatic dashpots;
3. electromagnetic dampers employing Foucault currents.

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Each of these types of dampers has its advantages and disadvantages.

The advantages and disadvantages of liquid or air-filled dashpots will be described below. Electromagnetic dampers employing eddy currents enjoy the advantage that, as in certain liquid-filled dashpots, the force with which they act on the

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damping mechanism is always proportional to the first power of the rate of motion of the part being damped, and is always in a direction opposite to that of the motion in question, i.e., this force F_d is equal to

$$F_d = -B\dot{x},$$

where B is a constant coefficient,

$\dot{x}$ is the rate of motion of the part in question.

It is further to be noted that electromagnetic dampers do not share the shortcoming of liquid-filled dashpots, in which the damping coefficient B usually undergoes very pronounced variations with changes in temperature. This is due to the fact that the viscosity of many liquids used in these dampers varies greatly with changes in temperature. In electromagnetic dampers, the variation in the coefficient B with changes in temperature will, of course, be considerably less pronounced.

However, a shortcoming of electromagnetic dampers is the fact that the force of damping with Foucault currents is quite small in cases of relatively low rates of motion of the part to be damped. Therefore, in instruments in which the required damping coefficient B is not very small, it is necessary to set up very powerful magnetic fields, which lead to extreme increases in the dimensions and weight of the damper itself. As a result, dampers of this type have not come into wide use.

The theory of such dampers is obviously of considerable interest, but in the present book we will consider only the fundamentals of the calculations as such; we will not discuss the theory of electromagnetic dampers but will deal only with dashpots.

The principle on which these dampers operate is that a displacement of the ^{STAT} body whose motion has to be damped, causes a piston to undergo corresponding motion in a cylinder. Usually this is attained by connecting either the piston or the cylinder in some way to the body in question, while connecting the cylinder (or, on the other hand, the piston) to the element relative to which damping of the displaced body

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takes place.

Often, in such dashpots, the cylinder is filled with liquid on both sides of the piston. As the piston is displaced, this liquid flows from one part of the cylinder to another through a tube. In order for the liquid to flow, some pressure drop across the two ends of the tube is needed. A similar pressure differential exists between the two sides of the piston, resulting in the creation of a force counteracting the displacement of the piston relative to the cylinder. However, this force is active only when some rate of displacement of the piston relative to the cylinder is operative. The greater the velocity, the higher the force. If there is no relative displacement, this force is equal to zero. Let us establish the relationship existing between the rate of displacement of the piston and the resultant force counteracting this displacement.

Liquid is able to flow from one part of the cylinder to another by two routes: a) through the connecting pipe and b) through the clearance between piston and cylinder. The flow of liquid per second through the tube can be determined by means of Poiseuille's equation (since, in the dashpots under consideration the flow is always laminar)

$$Q_1 = \frac{\pi r^4 \Delta p}{8 \eta l_1}, \quad (12.1)$$

where Q_1 is the flow of liquid;

r is the tube radius;

Δp is the pressure drop across the two ends of the tube (or, in other words, across the two sides of the piston);

l_1 is the length of the tube,

η is the unit coefficient of viscosity in gm-sec/cm^2 .

Let us derive the corresponding formula for the flow Q_2 through the gap. Since the ratio of the diameter of the piston to that of the cylinder is close to unity (the gap between piston and cylinder is usually quite small), let us consider the

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lengths of the two sides of the gap to be identical, for purposes of approximation. Examining a portion of this gap, let us imagine it to be bounded by two flat, non-cylindrical, walls.

Thus, let the length of the portion of the gap under examination be a (Fig.93); the distance between the walls (the annular gap between cylinder and piston), be δ ; and the depth of the gap (the length of the spring) be l_2 . As in the derivation of

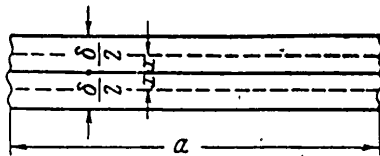


Fig.93 - See Derivation of Approximated Equation for Calculation of Velocity of Flow of Liquid Through an Annular Gap

Poiseuille's formula for tubes, let us consider the liquid immediately adjacent to the walls to be stationary, while, the further the particles of the liquid are from the walls, the greater their rate of displacement.

Let us consider the plane π_1 , at an identical distance $\frac{\delta}{2}$ from the two walls of the gap. Let us consider the volume of the liquid between two planes parallel to

the plane π_1 (meaning, therefore, the walls of the gap) and at a distance x from the plane π_1 , on both sides thereof. When this volume of liquid is in motion, its movement is resisted by a force F of viscous friction with the adjacent layers of liquid, which equals

$$F = -S\mu \frac{dv}{dx},$$

where S is the surface of contact between the layers,

v is the velocity of displacement of the layer.

A minus sign is used since the velocity v declines as x increases. In our case,

$$S = 2al_2,$$

where l_2 is the depth of the gap.

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Thus,

$$F = -2al_2\mu \frac{dv}{dx}$$

This resistance is overcome by the force resulting from the difference in pressure Δp between the two sides of the layer (at entry to an emergence from the gap). This force is equal to $\Delta p \cdot 2xa$, since the area across which the pressure acts is $2xa$.

Thus, we have

$$2xa \Delta p = -2al_2\mu \frac{dv}{dx},$$

or

$$\Delta p x dx = -l_2\mu dv.$$

The value of Δp does not depend upon x , so that

$$\begin{aligned} \Delta p \int_x^{\frac{\delta}{2}} x dx &= -l_2\mu \int_v^0 dv; \\ \Delta p \left(\frac{\delta^2}{8} - \frac{x^2}{2} \right) &= l_2\mu v; \\ v &= \frac{\Delta p}{\mu l_2} \left(\frac{\delta^2}{8} - \frac{x^2}{2} \right). \end{aligned} \quad (12.2)$$

Let us consider all the liquid in the aperture to be divided into layers of infinitesimal thickness dx , parallel to the planes W_1 . The flow of liquid per second dQ_2 through each such layer of infinitesimal thickness, will be

$$dQ_2 = a dx v = \frac{\Delta p a}{\mu l_2} \left(\frac{\delta^2}{8} - \frac{x^2}{2} \right) dx.$$

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As a result, the total discharge Q_2 through the aperture will be

$$Q_2 = \frac{\Delta p a}{\mu l_2} \int_{-\frac{\delta}{2}}^{+\frac{\delta}{2}} \left(\frac{\delta^2}{8} - \frac{x^2}{2} \right) dx = \frac{\Delta p a}{\mu l_2} \left(\frac{\delta^3}{8} - \frac{\delta^3}{24} \right) = \frac{\Delta p a \delta^3}{12\mu l_2}. \quad (12.3)$$

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In order to arrive at the total flow through the entire annular aperture, it is necessary to replace a in eq.(12.3) by the value $2\pi R_{cp}$, where R_{cp} is the mean radius between the radius of the piston and that of the cylinder (which differ very slightly). Thus, for the case we are considering, we have

$$Q_2 = \frac{\pi R_{cp}^3 \Delta p}{6\mu l_2} \quad (12.4)$$

Here we need only indicate the following: If it is considered that, in the case in question, we are dealing not with a flat but with an annular aperture, the form of the equation for the flow of liquid in terms of the pressure drop across the two sides of the aperture undergoes some modification.

However, as indicated below, this latter exact formula is little suited for practical use, and its employment cannot be recommended under any conditions.

However, in view of the fact that this formula is cited in certain sources, we will discuss the matter briefly. Let us begin with the derivation of the formula in question.

Figure 94 presents a cross section of the annular aperture between a circle with a radius of R_1 and a circle with a radius of R_2 . Let us consider, in the flowing liquid, a cylindrical circular surface, of a radius x (see broken line in Fig.94). Since the various cylindrical layers of the liquid flow at different speeds, the cylindrical surface in question is subject to the action of a force of viscous friction F_{Tp} whose value is

$$F_{Tp} = \mu \frac{dv}{dx} 2\pi x l_2$$

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(the symbols in the above equation being the same as in the preceding derivations), since the surface of a cylinder is $2\pi x l_2$.

Thus, the following forces affect the system within the given cylinder of a radius x :

1. The force of fluid friction F_{Tp} .

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2. The force resulting from the difference in pressure between the two sides of the aperture and the inside of the cylinder (cross-hatched in Fig. 9/4). This force is equal to the pressure drop Δp , multiplied by the area πx^2 of the base of the cylinder.

3. Some force P with which the inside (cross-hatched) part of the cylinder is held in place, since the liquid flowing through the aperture tends to entrain this central section.

Thus, we have the following equations for the forces acting on the system under consideration:

$$\mu \frac{dv}{dx} 2\pi x l_2 + \Delta p \pi x^2 - P = 0.$$

The signs in front of the individual terms in this equation have been assigned

in accordance with the following considerations:

If the value of $\frac{dv}{dx}$ is positive, i.e., if the velocity rises with an increase in the distance x

from the center of the cylinder, then the force $F_{\tau p}$ of viscous friction is exerted in the same direction as the compressive force $\Delta p \pi x^2$. The sign in front of the force P may be either one, since the direction of this force constitutes a specific magnitude of the force P , which is as yet unknown.

From this equation, we have

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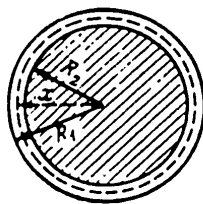


Fig. 9/4 - See Derivation of "Exact" Equation for Calculating the Rate of Flow of Liquid Through the Annular Aperture

$$-dv = \Delta p \frac{x dx}{2\mu l_2} - P \frac{dx}{2\pi \mu l_2 x}$$

From this, we obtain

$$-\int_v^0 dv = \frac{\Delta p}{2\mu l_2} \int_0^{R_1} x dx - \frac{P}{2\pi \mu l_2} \int_x^{R_1} \frac{dx}{x},$$

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since at $x = R_1$ (on the walls of the dashpot cylinder), the velocity v of the liquid is zero. Therefore,

$$v = \frac{\Delta p}{4\mu l_2} (R_1^2 - x^2) - \frac{P}{2\pi\mu l_2} \ln \frac{R_1}{x} \quad (12.5)$$

At $x = R_2$ (at the walls of the central, cross-hatched cylinder), the velocity v is also zero. Therefore, substituting the value R_2 in eq.(12.5) for x , we have

$$0 = \frac{\Delta p}{4\mu l_2} (R_1^2 - R_2^2) - \frac{P}{2\pi\mu l_2} \ln \frac{R_1}{R_2}$$

From this it follows that

$$P = \Delta p \frac{\pi (R_1^2 - R_2^2)}{2 \ln \frac{R_1}{R_2}}$$

Substituting the resultant expression in eq.(12.5), we obtain

$$v = \frac{\Delta p}{4\mu l_2} \left[(R_1^2 - x^2) - \ln \frac{x}{R_1} \frac{(R_1^2 - R_2^2)}{\ln \frac{R_2}{R_1}} \right] \quad (12.6)$$

The following is the amount of liquid dQ , flowing per second through an infinitesimally thin cylindrical layer, dx in thickness, of a radius x :

$$dQ = 2\pi x dx v = \frac{\Delta p \pi}{2\mu l_2} \left[(R_1^2 - x^2) x - \frac{(R_1^2 - R_2^2)}{\ln \frac{R_2}{R_1}} x \ln \frac{x}{R_1} \right] dx.$$

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The total flow Q through the entire annular aperture is

$$Q = \frac{\Delta p \pi}{2\mu l_2} \left[\int_{R_2}^{R_1} (R_1^2 - x^2) x dx - \frac{(R_1^2 - R_2^2)}{\ln \frac{R_2}{R_1}} \int_{R_2}^{R_1} x \ln \frac{x}{R_1} dx \right] =$$

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$$\begin{aligned}
&= \frac{\Delta p \pi}{2\mu l_2} \left\{ \frac{R_1^4}{2} - \frac{R_1^2 R_2^2}{2} - \frac{R_1^4}{4} + \frac{R_2^4}{4} - \right. \\
&\quad \left. - \frac{(R_1^2 - R_2^2)}{\ln \frac{R_2}{R_1}} \left[\frac{x^2}{2} \ln \frac{x}{R_1} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \right] \right\} = \\
&= \frac{\Delta p \pi}{2\mu l_2} \left\{ \frac{R_1^4}{4} - \frac{R_1^2 R_2^2}{2} + \frac{R_2^4}{4} - \frac{R_1^2 - R_2^2}{\ln \frac{R_2}{R_1}} \left[-\frac{R_2^2}{2} \ln \frac{R_2}{R_1} - \frac{R_1^2}{4} + \frac{R_2^2}{4} \right] \right\} = \\
&= \frac{\Delta p \pi}{8\mu l_2} \left\{ R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\ln R_1 - \ln R_2} \right\} = \\
&= \frac{\Delta p \pi}{8\mu l_2} (R_1^2 - R_2^2) \left\{ R_1^2 + R_2^2 - \frac{R_1^2 - R_2^2}{\ln R_1 - \ln R_2} \right\}. \quad (12.7)
\end{aligned}$$

Here it can be readily shown that eq.(12.4) is a first approximation of the equation (12.7).

Now let us examine the expression in braces in eq.(12.7)

$$\begin{aligned}
R_1^2 + R_2^2 - \frac{R_1^2 - R_2^2}{\ln \frac{R_1}{R_2}} &= \left(R_{cp} + \frac{\delta}{2} \right)^2 + \left(R_{cp} - \frac{\delta}{2} \right)^2 - \\
&\quad - \frac{2R_{cp}\delta}{2 \left(\frac{\delta}{2R_{cp}} + \frac{1}{3} \frac{\delta^3}{2^3 R_{cp}^3} + \frac{1}{5} \frac{\delta^5}{2^5 R_{cp}^5} + \dots \right)} = \\
&= 2R_{cp}^2 + \frac{\delta^2}{2} - \frac{2R_{cp}^2 \delta}{1 + \frac{1}{12} \frac{\delta^2}{R_{cp}^2} + \frac{1}{80} \frac{\delta^4}{R_{cp}^4} + \dots} = 2R_{cp}^2 + \frac{\delta^2}{2} - \\
&\quad - 2R_{cp}^2 \left(1 - \left(\frac{1}{12} \frac{\delta^2}{R_{cp}^2} + \frac{1}{80} \frac{\delta^4}{R_{cp}^4} + \dots \right) + \left(\frac{1}{12} \frac{\delta^2}{R_{cp}^2} + \frac{1}{80} \frac{\delta^4}{R_{cp}^4} + \dots \right)^2 - \right. \\
&\quad \left. - \left(\frac{1}{12} \frac{\delta^2}{R_{cp}^2} + \frac{1}{80} \frac{\delta^4}{R_{cp}^4} + \dots \right)^3 + \dots \right) = \\
&= \frac{\delta^2}{2} + \delta^2 \left(\frac{1}{6} + \frac{1}{72} \frac{\delta^2}{R_{cp}^2} + \frac{11}{7560} \frac{\delta^4}{R_{cp}^4} + \frac{107}{453600} \frac{\delta^6}{R_{cp}^6} + \dots \right) = \\
&= \frac{2}{3} \delta^2 \left(1 + \frac{1}{48} \frac{\delta^2}{R_{cp}^2} + \frac{11}{5040} \frac{\delta^4}{R_{cp}^4} + \frac{107}{302400} \frac{\delta^6}{R_{cp}^6} + \dots \right). \quad (12.8)
\end{aligned}$$

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As δ is usually only a small fraction of R_{mean} in value, it is obvious that the terms in the series in parentheses decline very rapidly. If, keeping this in mind, we take in first approximation only the first term in the series (i.e., unity), we have

$$Q = \frac{\Delta p \pi (R_1^2 - R_2^2)}{8\mu l_2} \frac{2}{3} \delta^2 = \frac{\Delta p \pi 2 R_{cp} \delta}{8\mu l_2} \frac{3}{2} \delta^2 = \frac{\Delta p \pi R_{cp} \delta^3}{6\mu l_2},$$

i.e., again eq.(12.4).

The above derivations show that, when eq.(12.4) is used, the relative error ϵ in determining the flow Q will be approximately equal to

$$\epsilon \approx \frac{1}{48} \frac{\delta^2}{R_{cp}^2}$$

(in reality, the error ϵ is only slightly greater than this value).

Thus, even when the ratio $\frac{\delta}{R_{\text{mean}}} = 0.1$ (which is often, of course, an exaggerated figure) the resultant error is slightly more than 0.02% of the measured value.

When the $\frac{\delta}{R_{\text{mean}}}$ ratio declines, ϵ diminishes rapidly and, in the usual range of values, the $\frac{\delta}{R_{\text{mean}}}$ ratio is entirely imperceptible.

As far as the precise equation (12.7) is concerned, it suffers from the shortcoming that the small value of the $\frac{\delta}{R_{\text{mean}}}$ ratio therein (which is always the case in dashpots) results in a subtraction of the difference between $R_1^2 + R_2^2$ and $\frac{R_1^2 - R_2^2}{\ln \frac{R_1}{R_2}}$, each of which is many times greater than their difference (in the dashpots under consideration their values are tens or even hundreds of thousands of times greater than this difference). Moreover, one of these values contains a STAT rithm. Therefore, in order to obtain a more or less accurate result, it is necessary that the logarithm be taken to a very precise value, to prevent the error in calculation from reaching several hundred percent.

Thus, while the use of the approximated eq.(12.4) results in a systematic er-

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ror, the value of this error usually does not exceed a few thousandths, and more often merely a few ten-thousandths, of a percent.

When using the exact formula [eq.(12.7)] we may have an accidental error due to an inaccuracy in the table of logarithms, but this accidental error may readily attain values exceeding manyfold the very value to be determined (the accuracy of logarithmic tables is limited by the number of decimals to which the logarithm is calculated).

In view of the foregoing, we recommend using only the approximated formula [eq.(12.4)].

It should be further noted that eqs.(12.4) and (12.7) are similar in form in the sense that in both we have $Q = c\Delta p$, where c is a constant depending on the parameters of the damper, and it is only for the constants in these two formulas that the expressions differ somewhat. This being the case, the use of one or another of these two formulas does not reflect on the general course of the subsequent derivations of the formulas for calculating dashpots. In our further derivation, we will use only the expression for c according to eq.(12.4).

Thus, in accordance with eqs.(12.1) and (12.4), we have the following expression for the total volume of liquid flowing, within a dashpot, from one part of the cylinder into the other:

$$Q = Q_1 + Q_2 = \Delta p \left[\frac{\pi r^4}{8\mu l_1} + \frac{\pi R^4}{8\mu l_2} \right] = \Delta p (k_1 + k_2), \quad (12.9)$$

where the coefficients $k_1 = \frac{\pi r^4}{8\mu l_1}$ and $k_2 = \frac{\pi R^4}{8\mu l_2}$ depend on the designed dimensions of the dashpot. STAT

On the other hand, the amount of liquid flowing from one part of the cylinder into the other, is

$$Q = S_1 \frac{dy}{dt}$$

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where S_1 is the cross-sectional area of the cylinder,

$\frac{dy}{dt}$ is the rate of displacement of the piston in the cylinder.

Thus,

$$\Delta p (k_1 + k_2) = S_1 \frac{dy}{dt}, \quad (12.10)$$

from which

$$\Delta p = \frac{S_1}{k_1 + k_2} \frac{dy}{dt}.$$

The damping force F_d is

$$F_d = \Delta p S_2,$$

where S_2 is the area of the piston.

Thus,

$$F_d = \frac{S_1 S_2}{k_1 + k_2} \frac{dy}{dt}.$$

Since, usually, the radii of piston and cylinder differ little, it may be assumed, approximately, that

$$S_1 S_2 = (\pi R^2)^2$$

and

$$F_d = \frac{(\pi R^2)^2}{k_1 + k_2} \frac{dy}{dt}. \quad (12.11)$$

Thus, the value $\frac{(\pi R^2)^2}{k_1 + k_2}$ is the coefficient of damping B, contained in the equations of motion of the mechanisms. STAT

Therefore, the following procedure may be used in calculating a dashpot.

First, on the basis of an analysis of the dynamic errors of the mechanism, we arrive at the required damping coefficient B. Then we verify whether the value $\frac{(\pi R^2)^2}{k_2}$ is adequately larger than the required value of B (a condition that is usually easy to satisfy).

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Then, if the dashpot has no provision for regulating the value of k_1 , we must determine what this magnitude has to be in order to satisfy the equation

$$\frac{(\pi R^2)^2}{k_1 + k_2} = B,$$

i.e.,

$$k_1 = \frac{(\pi R^2)^2}{B} - k_2. \quad (12.12)$$

However, many dashpots do contain provisions for regulating the value of k_1 . Often a cock of some sort is used, which makes it possible to increase the resistance to the flow of liquid from one chamber of the cylinder to the other, through the tube. In this case it is necessary, in calculating the parameters of the tube, to base the calculation on the assumption that the cock is wide open, using the following equation

$$\frac{(\pi R^2)^2}{k_1 + k_2} = \frac{B}{n},$$

from which we obtain

$$k_1 = \frac{n (\pi R^2)^2}{B} - k_2, \quad (12.13)$$

where n is an assurance factor since, in the case in which dashpot adjustment is available, it must be remembered that damping is at its lowest at a wide open cock. The determination of the magnitude of the coefficient n depends on the assumed range of damping adjustment. Often this coefficient may be calculated to be 2 or 3.

Having determined the required value of k_1 from eqs.(12.13) or (12.12), we may use eq.(12.1) to determine the required parameters of the tube. STAT

In certain cases of calculation of dashpots, Q_2 may be represented by an expression somewhat different from eqs.(12.4) and (12.9). In fact, it is sometimes more correct to assume that the piston is pressed against one side of the cylinder (this is the case, for example, when the dashpot is in a horizontal position).

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In this case, at an approximated derivation of the equation for calculating the value of Q_2 , we take into consideration the proofs derived above to the effect that the use of eq.(12.4) results only in imperceptible calculation errors as well as the fact that, in dashpots, the $\frac{\delta}{R_{\text{mean}}}$ ratio is usually very small.

On the basis of these considerations, let us assume, in first approximation, that the clearance δ_1 between piston and cylinder, varying around the circumference of the cylinder, can be expressed approximately as follows:

$$\delta_1 \approx \delta(1 - \cos \varphi),$$

where φ is the angle between the radius of the cylinder at the contact point of piston and cylinder and the radius along which the value of the clearance δ_1 is determined.

Further, let us consider that

$$\begin{aligned} Q_2 &\approx \frac{\Delta p R_{cp}}{12\mu l_2} \int_0^{2\pi} \delta_1^3 d\varphi = \frac{R_{cp} \delta^3 \Delta p}{12\mu l_2} \int_0^{2\pi} (1 - \cos \varphi)^3 d\varphi = \\ &= \frac{R_{cp} \delta^3 \Delta p}{12\mu l_2} \left[\int_0^{2\pi} d\varphi - 3 \int_0^{2\pi} \cos \varphi d\varphi + 3 \int_0^{2\pi} \cos^2 \varphi d\varphi - \int_0^{2\pi} \cos^3 \varphi d\varphi \right] = \\ &= \frac{R_{cp} \delta^3 \Delta p}{12\mu l_2} \left[2\pi + 3 \int_0^{2\pi} \cos^2 \varphi d\varphi \right] \end{aligned}$$

and since $\int_0^{2\pi} \cos^2 \varphi d\varphi = \int_0^{2\pi} \sin^2 \varphi d\varphi$, a $\int_0^{2\pi} \cos^2 \varphi d\varphi + \int_0^{2\pi} \sin^2 \varphi d\varphi = 2\pi$,

$$Q_2 \approx \frac{R_{cp} \delta^3 \Delta p}{12\mu l_2} [2\pi + 3\pi] = \frac{5\pi R_{cp} \delta^3 \Delta p}{12\mu l_2}.$$

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In this case, instead of eq.(12.11) we will have

$$Q = Q_1 + Q_2 = \Delta p \left[\frac{\pi r^4}{8\mu l_1} + \frac{5\pi R_{cp} \delta^3}{12\mu l_2} \right] = \Delta p (k_1 + k_2),$$

where

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$$k_1 = \frac{\pi r^4}{8\mu l_1} \text{ and } k_2 = \frac{5\pi R^3}{12\mu l_2}$$

The rest of the calculation for the dashpot will not differ from that described above.

Dashpots may also be pneumatic. In such cases, damping will have a somewhat different character. Let us analyze this case.

In the derivation of eqs. (12.1), (12.4), and (12.7), the liquid was regarded as incompressible. In the case of a gas (a compressible fluid), the above derivations permit us to develop only the following equations for the volume flow of gas through the tube and through the clearance:

$$Q_1 = \frac{\pi r^4 dp}{8\mu dl_1}$$

and

$$Q_2 = \frac{\pi R^3 dp}{6\mu dl_2}$$

where dl_1 is an infinitesimal portion of the total length l_1 of the tube;

dl_2 is an infinitesimal portion of the total length l_2 of the piston;

dp is the pressure drop across the infinitesimally small length dl_1 or dl_2 .

Only across such an infinitesimal length can the density of the gas be regarded as constant. But in the case of stabilized motion, we may consider the flow of gas by weight per second as constant across any cross section of the channel. Thus, we may write for the tube (for its total length l_1)

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$$\frac{dP_1}{dt} = Q_1 \gamma = \frac{\pi r^4 dp}{8\mu dl_1} \gamma = \text{const},$$

and for the annular gap (for the entire length l_2 of the piston)

$$\frac{dP_2}{dt} = Q_2 \gamma = \frac{\pi R^3 dp}{6\mu dl_2} \gamma = \text{const},$$

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where ρ is the density of the gas by weight in the cross section under examination,
and $\frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ represent the gas consumption by weight.
Thus, we have

$$\left. \begin{aligned} \frac{\pi r^4}{8\mu} \gamma dp &= \frac{dP_1}{dt} dl_1; \\ \frac{\pi R \delta^3}{6\mu} \gamma dp &= \frac{dP_2}{dt} dl_2. \end{aligned} \right\} \quad (12.14)$$

However,

$$\gamma = \frac{p}{R_{\text{gas}} T},$$

where p is the gas pressure,

R_{gas} is the constant of the gas,

T is the absolute temperature of the gas.

Thus,

$$\frac{\pi r^4 p dp}{8 R_{\text{gas}} \mu T} = \frac{dP_1}{dt} dl_1$$

and

$$\frac{\pi R \delta^3 p dp}{6 R_{\text{gas}} \mu T} = \frac{dP_2}{dt} dl_2.$$

In a gas of isothermal flow (in which the gas is able to acquire the temperature of its environment through the walls of the channel, as it flows), T and μ are constant, so that we will have

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$$\frac{\pi r^4}{8 R_{\text{gas}} \mu T} \int_{P_1}^{P_2} p dp = \frac{dP_1}{dt} \int_0^{l_1} dl_1$$

and

$$\frac{\pi R \delta^3}{6 R_{\text{gas}} \mu T} \int_{P_1}^{P_2} p dp = \frac{dP_2}{dt} \int_0^{l_2} dl_2.$$

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where p_2 and p_1 are the pressures at the two ends of the channel or aperture, i.e.,

$$p_2 - p_1 = \Delta p.$$

Thus,

$$\frac{\pi r^4 (p_2^2 - p_1^2)}{16 R_{gas} \mu T} = \frac{dP_1}{dt} l_1$$

and

$$\frac{\pi R \delta^3 (p_2^2 - p_1^2)}{12 R_{gas} \mu T} = \frac{dP_2}{dt} l_2$$

or

$$\left. \begin{aligned} \frac{dP_1}{dt} &= \frac{\pi r^4 (p_2^2 - p_1^2)}{16 R_{gas} \mu T l_1} = \frac{\pi r^4 \Delta p}{8 \mu l_1} \frac{p_2 + p_1}{2 R_{gas} T} \\ \frac{dP_2}{dt} &= \frac{\pi R \delta^3 (p_2^2 - p_1^2)}{12 R_{gas} \mu T l_2} = \frac{\pi R \delta^3 \Delta p}{6 \mu l_2} \frac{p_2 + p_1}{2 R_{gas} T} \end{aligned} \right\} \quad (12.15)$$

i.e., the flow of gas by weight is equal to its flow by volume in accordance with eqs.(12.1) and (12.4), multiplied by the average density of the gas at the two ends of the channel or aperture.

If the process is adiabatic, the values of T and μ are variable, since the temperature T is functionally related to the pressure p , while the viscosity μ is a function of the temperature T . For adiabatic processes, we have the equation

$$\left(\frac{p}{p_0} \right) = \left(\frac{\gamma}{\gamma_0} \right)^k \quad (12.16)^{STAT}$$

(for air, $k = 1.41$).

Here γ is the density of the gas by weight,

p_0 is a constant pressure,

γ_0 is the density of the gas by weight, corresponding to this pressure.

Let us take p_0 as equal to some average pressure in the tube (or aperture).

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From this equation, we have

$$\frac{dp}{p} = k \frac{d\gamma}{\gamma},$$

or

$$dp = k \frac{p d\gamma}{\gamma}. \quad (12.17)$$

Substituting this expression in eq.(12.14), we have

$$\left. \begin{aligned} \frac{\pi r^4}{8\mu} k p d\gamma &= \frac{dP_1}{dt} dl_1; \\ \frac{\pi R^3}{6\mu} k p d\gamma &= \frac{dP_2}{dt} dl_2. \end{aligned} \right\} \quad (12.18)$$

The viscosity of air is almost proportional to its absolute temperature, according to tabular data. Therefore, for our case, in which the temperature fluctuations occurring as a result of the adiabatic process, are not very great, we may consider, for the sake of simplicity, that

$$\mu \approx \mu_0 \frac{T}{T_0},$$

where T_0 is some average air temperature in the channel at the stabilized rate of flow,

μ_0 is the corresponding viscosity of the air.

Substituting this equation in eq.(12.18), and multiplying the numerator and denominator on the left side of the equation by R_{gas} , we obtain

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$$k \frac{\pi r^4 T_0 R_{\text{gas}}}{8\mu_0} \frac{p d\gamma}{R_{\text{gas}} T} = k \frac{\pi r^4 T_0 R_{\text{gas}}}{8\mu_0} \gamma d\gamma = \frac{dP_1}{dt} dl_1$$

and

$$k \frac{\pi R^3 T_0 R_{\text{gas}}}{6\mu_0} \frac{p d\gamma}{R_{\text{gas}} T} = k \frac{\pi R^3 T_0 R_{\text{gas}}}{6\mu_0} \gamma d\gamma = \frac{dP_2}{dt} dl_2.$$

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Integrating these equations, we obtain

$$k \frac{\pi r^4 T_0 R_{gas}}{8\mu_0} \frac{\gamma_2^2 - \gamma_1^2}{2} = \frac{dP_1}{dt} l_1$$

and

$$k \frac{\pi R^3 T_0 R_{gas}}{6\mu_0} \frac{\gamma_2^2 - \gamma_1^2}{2} = \frac{dP_2}{dt} l_2$$

from which

$$\left. \begin{aligned} \frac{dP_1}{dt} &= \frac{k\pi r^4 T_0 R_{gas}}{8\mu_0 l_1} \frac{(\gamma_2 - \gamma_1)(\gamma_2 + \gamma_1)}{2} \\ \text{and} \quad \frac{dP_2}{dt} &= \frac{k\pi R^3 T_0 R_{gas}}{6\mu_0 l_2} \frac{(\gamma_2 - \gamma_1)(\gamma_2 + \gamma_1)}{2} \end{aligned} \right\} \quad (12.19)$$

In accordance with eq.(12.16), we have

$$\gamma_1 = \gamma_0 \left(\frac{p_1}{p_0} \right)^{\frac{1}{k}} \quad \text{and} \quad \gamma_2 = \gamma_0 \left(\frac{p_2}{p_0} \right)^{\frac{1}{k}}$$

On the basis of this expression, eq.(12.19) may be written as follows:

$$\frac{dP_1}{dt} = \frac{k\pi r^4 T_0 R_{gas}}{8\mu_0 l_1} \frac{\gamma_1 + \gamma_2}{2} \frac{\gamma_0}{p_0} p_0^{1-\frac{1}{k}} \left(p_2^{\frac{1}{k}} - p_1^{\frac{1}{k}} \right)$$

and

$$\frac{dP_2}{dt} = \frac{k\pi R^3 T_0 R_{gas}}{6\mu_0 l_2} \frac{\gamma_1 + \gamma_2}{2} \frac{\gamma_0}{p_0} p_0^{1-\frac{1}{k}} \left(p_2^{\frac{1}{k}} - p_1^{\frac{1}{k}} \right)$$

and as $\gamma_0 = \frac{p_0}{T_0 R_{gas}}$, thus

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$$\left. \begin{aligned} \frac{dP_1}{dt} &= \frac{k\pi r^4}{8\mu_0 l_1} \frac{\gamma_1 + \gamma_2}{2} p_0^{1-\frac{1}{k}} \left(p_2^{\frac{1}{k}} - p_1^{\frac{1}{k}} \right) \\ \text{and} \quad \frac{dP_2}{dt} &= \frac{k\pi R^3}{6\mu_0 l_2} \frac{\gamma_1 + \gamma_2}{2} p_0^{1-\frac{1}{k}} \left(p_2^{\frac{1}{k}} - p_1^{\frac{1}{k}} \right) \end{aligned} \right\} \quad (12.20)$$

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In accordance with the theorem of finite differences, we have

$$\frac{1}{p_2^k} - \frac{1}{p_1^k} = (p_2 - p_1) \frac{1}{k} p_{\text{mean}}^{\frac{1}{k}-1} = \Delta p \frac{1}{k} p_{\text{mean}}^{\frac{1}{k}-1},$$

where p_{mean} is a mean pressure between p_1 and p_2 .

Substituting this equation in eq.(12.20), we obtain

$$\frac{dP_1}{dt} = \frac{\pi r^4 \Delta p}{8\mu_0 l_1} \frac{\gamma_1 + \gamma_2}{2} p_0^{1-\frac{1}{k}} p_{\text{mean}}^{\frac{1}{k}-1}$$

and

$$\frac{dP_2}{dt} = \frac{\pi R^4 \Delta p}{6\mu_0 l_2} \frac{\gamma_1 + \gamma_2}{2} p_0^{1-\frac{1}{k}} p_{\text{mean}}^{\frac{1}{k}-1}$$

If p_0 is assumed as approximately the same as p_{mean} , we have

$$\left. \begin{aligned} \frac{dP_1}{dt} &\approx \frac{\pi r^4 \Delta p}{8\mu_{\text{mod}1}} \frac{\gamma_1 + \gamma_2}{2} \\ \frac{dP_2}{dt} &\approx \frac{\pi R^4 \Delta p}{6\mu_{\text{mod}2}} \frac{\gamma_1 + \gamma_2}{2} \end{aligned} \right\} \quad (12.21)$$

i.e., with the above simplification, to the effect that the viscosity of air is approximately proportional to its absolute temperature, we obtain, for an adiabatic flow, approximately the same equation as for isothermal flow.

Thus, we are always able to assume that, approximately, the total flow of air through the channel and the annular aperture is

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$$\left. \begin{aligned} \frac{dP}{dt} &= \frac{dP_1}{dt} + \frac{dP_2}{dt} \approx \Delta p \frac{\gamma_1 + \gamma_2}{2} \frac{\pi}{\mu_{\text{mean}}} \left(\frac{r^4}{8l_1} + \frac{R^4}{6l_2} \right) = c \Delta p \frac{\gamma_1 + \gamma_2}{2}, \\ \text{where} \quad c &\approx \frac{\pi}{\mu_{\text{mean}}} \left(\frac{r^4}{8l_1} + \frac{R^4}{6l_2} \right) \end{aligned} \right\} \quad (12.22)$$

is a constant for the given dashpot.

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Pneumatic dashpots are usually designed so that one side of the cylinder is open, and air from the other portion is expelled into the atmosphere as the piston approaches the bottom of the cylinder and, conversely, intake of air occurs as the piston moves in the opposite direction.

If we always use Δp to designate excess pressure within the cylinder, measured against the atmospheric pressure, i.e., if the pressure $p_2 = p_1 + \Delta p$ is always the pressure within the cylinder and p_1 is the external pressure (so that Δp may be either positive or negative), and if P designates the weight of air in the cylinder, eq. (12.22) assumes the form

$$\frac{dP}{dt} = -c \frac{\gamma_1 + \gamma_2}{2} \Delta p, \quad (12.23)$$

since, at positive Δp , there will be an outflow of air and, at a negative value, air will be taken into the cylinder.

On the other hand, the weight of air in the cylinder will be

$$P = V \gamma_2,$$

where V is the volume of air.

On this basis, we have

$$\frac{dP}{dt} = \frac{dV}{dt} \gamma_2 + V \frac{d\gamma_2}{dt}. \quad (12.24)$$

Comparing eqs. (12.23) and (12.24), we obtain

$$-c \frac{\gamma_1 + \gamma_2}{2} \Delta p = \frac{dV}{dt} \gamma_2 + V \frac{d\gamma_2}{dt}. \quad \text{STAT}$$

However, $V = S_1 y$, where S_1 is the cross-sectional area of the cylinder and y is the distance (in some cases, the reduced distance) between the piston and the cylinder bottom. In addition,

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$$\frac{\gamma_1 + \gamma_2}{2} = \gamma_2 - \frac{\gamma_2 - \gamma_1}{2} = \gamma_2 - \frac{\Delta\gamma}{2},$$

where $\Delta\gamma = \gamma_2 - \gamma_1$.

Therefore,

$$-c\gamma_2\Delta p + \frac{c}{2}\Delta\gamma\Delta p = S_1\gamma_2\frac{dy}{dt} + S_1y\frac{d\gamma_2}{dt},$$

or

$$c\Delta p + \frac{c}{2}\Delta p\frac{\Delta\gamma}{\gamma_2} = S_1\frac{dy}{dt} + S_1y\frac{d\gamma_2}{dt}. \quad (12.25)$$

Thus we have arrived at an equation differing from eq.(12.10) in that it includes the additional terms $\frac{c}{2}\Delta p\frac{\Delta\gamma}{\gamma_2}$ and $S_1y\frac{d\gamma_2}{dt}$.

Note: The fact that the sign of the term $c\Delta p$ is different from that in eq.(12.10) is of no significance in the given instance, since this change in sign is a result only of the fact that, in deriving eq.(12.25), we examined the amount of gas on a specific side of the piston, which was not the case in the derivation of eq.(12.10).

The presence of these additional terms $\frac{c}{2}\Delta p\frac{\Delta\gamma}{\gamma_2}$ and $S_1y\frac{d\gamma_2}{dt}$ results in a significant difference in the nature of pneumatic piston damping from that of liquid damping with an identical piston. The result is that the nature of pneumatic damping makes it quite inconvenient [as may readily be seen from a study of eq.(12.25)] for solving equations for the motion of bodies, in which this type^{STAT} damping occurs. This is also comprehensible from the view point of physics since, in a pneumatic dashpot, not only does the air in the cylinder fail to inhibit the motion of the cylinder in the first instants after change in direction but it even assists this motion, since the cylinder is already filled with air, compressed or rarified, depending on the preceding motion of the piston in the opposite direction.

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Thus, a pneumatic dashpot, during certain intervals, operates more as an additional spring in the system than as a dashpot. It would appear that this is particularly evident when the vibrations of the body whose motions are being damped, have a high frequency.

A pneumatic dashpot works in a similar fashion when the piston undergoes sharp and rapid changes in direction, and a very insignificant amount of air is able to pass through the capillary tube and the annular gap by which the interior of the dashpot connects with the atmosphere. This last consideration also follows from eqs.(12.25). The fact is that, in this case, the left side of eq.(12.25) may be taken as zero, so that we obtain

$$\frac{dy}{dt} \approx -\frac{d\gamma_2}{\gamma_2},$$

and on the basis of eq.(12.17) we have

$$\frac{dy}{dt} \approx -\frac{1}{k} \frac{dp_2}{p_2}.$$

Integrating this equation, we obtain

$$\ln \frac{y}{y_0} = \frac{1}{k} \ln \frac{p_1}{p_2},$$

where y_0 is a value for the distance y , at which the pressure underneath the piston is equal to the outside pressure.

As a result,

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$$\frac{p_2}{p_1} = \left(\frac{y_0}{y}\right)^k$$

and

$$F_d = S_1 \Delta p = S_1 (p_2 - p_1) = S_1 p_1 \left[1 - \left(\frac{y_0}{y}\right)^k\right].$$

Thus, in the case under examination, the value of F_d is a function of the dis-

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tance y between the piston and the cylinder bottom rather than a function of the rate of change in this distance, i.e., the force created by the piston is elastic in nature.

In addition as soon as the body undergoes oscillation of some kind, the damping will depend not only on the rate of motion of the piston but also on the amplitude of displacement thereof since, in order for the air in the cylinder to exert pressure on the piston resisting its motion, the air must undergo compression, i.e., a reduction in volume. Therefore, if the amplitude of vibration is low, the damping effect may be quite insignificant, even if the frequency is high.

It is clear from the foregoing that pneumatic dashpots are preferably used for damping systems not subject to vibrations of particularly high frequency. Thus, this type of dashpot is used in gyro turn indicators, since the period of natural oscillation in such instruments is of the order of 0.5 sec, while the period during which the aircraft flutters (the perturbation force) is considerably greater.

It is clear from eq.(12.11) that liquid dashpots are quite suited to their purpose in that they set up a damping force normal to the first power of the velocity. This, as we know, is also very handy for solving differential equations of motion for the body with which such a dashpot is mounted. But it must be noted that such dashpots have the disadvantage that variations in temperature result in a pronounced change in the coefficient of viscosity of virtually all liquids used therein; this means that the degree of damping, when using such a damper, is reduced by the same factor. In this connection, gases have the advantage that their viscosity changes comparatively little with variations in temperature.

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Equation (12.25) is, of course, quite clumsy with respect to the derivation of the value of Δp therefrom as a function of the value y of the displacement of the piston and its derivatives.

Therefore, an analysis of this type of pneumatic damper by means of eq.(12.25) is quite difficult, and it is more accurate to state that often it is even impossi-

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ble. In view of this, dashpots of this type, in very many cases, are calculated in advance, in very rough approximation, by means of eqs.(12.1) and (12.4) for liquid dashpots, or in other words, the compressibility of air is disregarded.

Of course, a calculation of pneumatic dashpots by the formulas for liquid dashpots often proves highly inaccurate.

For example, if, by calculation, the value of B is large enough (i.e., if resistance in the outlet capillary is sufficient), and the oscillation frequency ω of the piston is adequate, a pneumatic dashpot will obviously function more as an auxiliary spring than as a dashpot (naturally, the forces it creates on oscillations of the piston will not be in phase with the calculated oscillations of the forces to be damped: they may actually be more nearly in phase with the vibrations of the forces from elastic elements in the mechanism).

In the present work, we have limited ourselves solely to an examination of components of instrument mechanisms, and have not discussed problems having to do with the design and calculation of various other components (such as bearings, guides for mounting parts, winding springs, trigger controls, etc.) since analyses and descriptions of such parts have been given in great detail in a number of sources on instrument parts (O.Rikhter and R.Foss, F.V.Drozдов, et al).

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Page	Line	Heads	Should read	Responsible
74	10 from bottom	$\delta = -\frac{1}{2}(\bar{\gamma} - \bar{\beta})$	$\delta = -\frac{1}{2}(\bar{\gamma} - \bar{\beta})$	publisher's proofreader
105	3 from bottom	moment N_1	moment M	author
150	4 from bottom	$-\frac{d\epsilon d\gamma}{\cos \alpha}$	$-\frac{d\epsilon d\gamma}{2 \cos \alpha}$	publisher's proofreader
167	2 from top	$-\frac{EJ}{2}$	$-\frac{EJ}{2}$	printer
167	2 from bottom	$\frac{EJ}{2(3L-1)}$	$\frac{EJ}{2(3L-1)}$	printer
245	3 from bottom	$+\frac{E}{3}$	$+\frac{E}{3}$	author
323	3 from top	with the following component	with the preceding component	author
325	2nd and 3rd from bottom	$-\frac{1}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1}} \frac{dl}{dz}$	$-\frac{1}{N_2 \sqrt{\left(\frac{L}{z}\right)^2 - 1 - N_2}} \frac{dl}{dz}$	author
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		$\frac{n_1 N_2 \left\{ \left[\left(\frac{L}{z} \right)^2 - 1 \right] - \frac{1}{L} \left(\frac{L}{z} \right)^3 \right\} - n_1 N_2}{\sqrt{\left(\frac{L}{z} \right)^2 - 1} \left(N_2 \sqrt{\left(\frac{L}{z} \right)^2 - 1 - N_2} \right)^2}$	$\frac{n_1 N_2 \left\{ \left[\left(\frac{L}{z} \right)^2 - 1 \right] - \frac{1}{L} \left(\frac{L}{z} \right)^3 \right\} + n_1 N_2}{\sqrt{\left(\frac{L}{z} \right)^2 - 1} \left(N_2 \sqrt{\left(\frac{L}{z} \right)^2 - 1 - N_2} \right)^2}$	
326	5 from top	$R = \frac{n_2}{N_2 \text{cig } \gamma - N_2}$	$R = \frac{n_2}{N_2 \text{cig } \gamma - N_2} +$	author
		$\frac{n_1 N_2 \left(\text{cig}^2 \gamma - \frac{1}{L \sin^2 \gamma \cos \gamma} \right) - n_1 N_2 \text{ig } \gamma}{(N_2 \text{cig } \gamma - N_2)^2}$	$\frac{n_1 N_2 \left(\text{cig}^2 \gamma - \frac{1}{L \sin^2 \gamma \cos \gamma} \right) + n_1 N_2 \text{ig } \gamma}{(N_2 \text{cig } \gamma - N_2)^2}$	

* Translator's note: The page numbers refer to original book; all corrections have been made.

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